Scrambling a Rubik's Cube

Hua Zhou

February 24, 2006

1 Introduction

A normal ($3 \times 3 \times 3$) Rubik's cube has six faces, denoted by u(p), d(own), l(eft), r(ight), f(ront), and b(ack) respectively. Each face has 9 facets. The basic moves are U, D, L, R, F, and B, which rotate the corresponding face 90 degrees clockwise (facing that face). In all moves, positions of the 6 center facets are fixed.

At the *solved position*, each of the 6 faces has same color. Goal of the puzzle is to return the cube, after certain number of randomized repeated rotations, to the solved position. Here we are concerned with sort of converse of the puzzle, that is how many rotations are necessary to bring the cube to a random position.

2 Mathematical Description of $3 \times 3 \times 3$ Rubik's Cube and A Geometric Bound

Let G_3 be the $3 \times 3 \times 3$ *Rubik's cube group* - the group of all possible (legal) moves generated from the 6 basic moves, i.e., $G_3 =< U, D, L, R, F, B >$. As shown in Joyner [8], G_3 is a normal subgroup of $H_3 \cong (C_3 \wr S_8) \times (C_2 \wr S_{12})$ of index 12. H_3 is the group of all (legal or illegal) moves of the Rubik's cube. Consider action of H_3 on the cube. Each move $(\vec{v}, \rho, \vec{w}, \sigma) \in C_3^8 \times S_8 \times C_2^{12} \times S_{12}$ in H_3 (1) permutes the 8 corner subcubes by $\rho \in S_8$ and twists them independently by $\vec{v} \in C_3^8$ and (2) permutes the 12 edge subcubes by $\sigma \in S_{12}$ and flips them independently by $\vec{w} \in C_2^{12}$. Then the $3 \times 3 \times 3$ Rubik's cube group G_3 contains those moves in H_3 that satisfy

- $\sum_{i=1}^{8} v_i = 0$ ("conservation of total twists");
- $\sum_{i=1}^{12} w_i = 0$ ("conservation of total flips");
- $\text{sgn}(\rho)\text{sgn}(\sigma) = 1$ ("equal parity as permutations").

So the size of the Rubik's cube group is $|G_2| = |H|/12 = 3^8 8! 2^{12} 12!/12 = 43,252,003,274,489,856,000 \approx 4.3 \times 10^{19}$.

For the random walk K on G_3 driven by p uniform on $S = \{\mathrm{id}, U^{\pm 1}, D^{\pm 1}, L^{\pm 1}, R^{\pm 1}, F^{\pm 1}, B^{\pm 1}\}$, we want to know how many steps are required to bring the cube into a random position (stationary distribution is uniform $\pi = 1/|G_3|$). p is symmetric, let $1 = \beta_0 \geq \beta_1 \geq \cdots \geq \beta_{|G_3|-1}$ be the real eigenvalues of the associated operator $L^2(G_3)$.

We apply the geometric method to get a first bound on the second largest eigenvalue β_1.

Proposition 1. *Assume that $\mathcal{X} = G$ is a finite group with generating set $S = \{g_1, \ldots, g_s\}$. Set $K(x,y) = |S|^{-1} 1_S(x^{-1}y), \pi \equiv 1/|G|$. Then $\beta_1 \leq 1 - \frac{1}{2|S|D^2}$ where D is the diameter of the Cayley graph $(G, S \cup S^{-1})$. If S is symmetric, i.e., $S = S^{-1}$, then $\beta_1 \leq 1 - \frac{1}{|S|D^2}$.*

Proof. See, e.g., Saloff-Coste (1996) [11] p378. □

Corollary 2. *For simple random walk on Cayley graph (G_3, S) (G_3 the $3 \times 3 \times 3$ Rubik's cube group, S the idendity and 12 quarter turns),*

$$\beta_1 \leq 1 - \frac{1}{13 \times 42^2}.$$

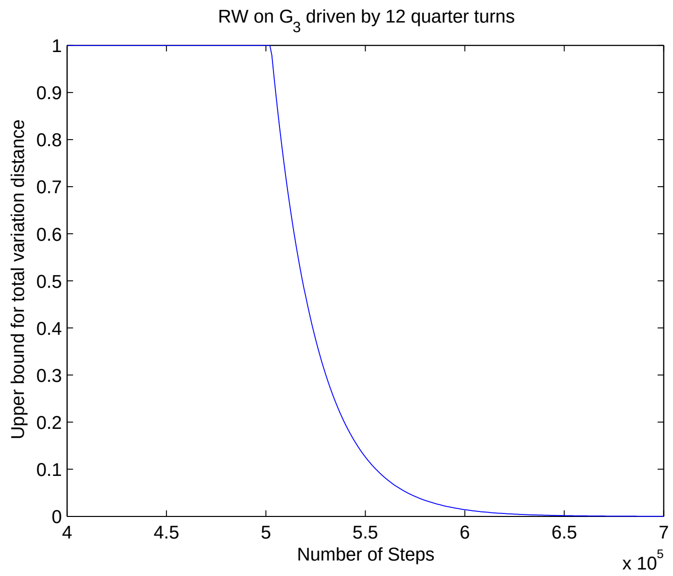
Proof. S is symmetric. The diameter D of Cayley graph (G_3, S) for Rubik's cube group is unknown. But we know it is bounded above by 42. See, e.g., Weisstein [15]. □

Combining with the bound for the smallest eigenvalue $\beta_{|G_3|-1} \geq -1 + p(\mathrm{id}) = -12/13$. $\beta^* = \max\{\beta_1, |\beta_{|G_3|-1}|\} \leq 1 - 1/(13 \times 42^2)$ and thus (by Upper Bound Lemma, Proposition 21)

$$\|K^{(n)} - \pi\|_{TV} \leq \frac{1}{2}|G_3|^{1/2} \beta^{*n}. \tag{1}$$

This implies that > half million steps are enough to make a $3 \times 3 \times 3$ Rubik's cube random.

Remark: If a "supercubist" can do 10 quarter turns per second, it will take him/her 13.89 hours to do half million quarter turns!



It is desirable to do complete eigen-analysis of some random walks driven by conjugacy classes in the Rubik's cube group. Structure of $3 \times 3 \times 3$ Rubik's cube group suggests that we may first consider two sub-problems, i.e., the groups of moves of the 8 corner subcubes and 12 edge subcubes respectively and then combine to get insights into the whole $3 \times 3 \times 3$ Rubik's cube group.

3 Corner Subcubes of a $3 \times 3 \times 3$ Rubik's Cube - $2 \times 2 \times 2$ Rubik's Cube (Pocket Cube)

We might think of a $2 \times 2 \times 2$ Rubik's cube (also called a Pocket Cube) as the normal $3 \times 3 \times 3$ Rubik's cube with the same moves but without all the 12 edge subcubes. Let G_V denote the group of all (legal) moves of a $2 \times 2 \times 2$ Rubik's Cube. Then similar proof shows that G_V is a subgroup of $H_V \cong C_3 \wr S_8$ of index 3, i.e., $G_V \cong C_3^7 \rtimes S_8$ contains all moves in H_V that conserve the total twists. The size of G_V is $|G_V| = 3^8 8!/3 = 88,179,840 \approx 8.8 \times 10^7$.

Remark: 1. For a real $2 \times 2 \times 2$ Rubik's cube, there is nothing identifying the orientation of the cube, reducing the number of positions by a factor of 24. But the way we think about it (just the 8 corner subcubes of a normal Rubik's cube) implies that the "imaginary center faces" still gives the orientation. We insist this to make it easier to study the normal $3 \times 3 \times 3$ Rubik's cube later. 2. The action of S_8 on C_3^7 is clear in this context. We will omit the action symbol when writing the semi-direct product $C_m^{n-1} \rtimes S_n$.

3.1 Conjugacy classes and irreducible characters of G_V

Chapter 4 of James and Kerber (1981) [6] discusses the conjugacy classes and irreducible representations of wreath product $G \wr S_n$. In order to get character table of $G_V = C_3^7 \rtimes S_8$, we need to figure out how the conjugacy classes and irreducible characters of $C_3 \wr S_8$ split in this subgroup.

For each element $(\vec{v}, \rho)$ in the wreath product $C_m \wr S_n$, we can write down a m-by-n matrix $\{a_{ij}\}, 0 \le i \le m-1, 1 \le j \le n$, where a_{ij} is the number of j-cycles (in cycle decomposition of ρ) with cycle product i. (*Cycle product* is the product of elements in $\vec{v}$ indexed by that cycle.) The matrix $a(\vec{v}, \rho) := \{a_{ij}\}$ thus defined is called the *type-matrix* of the group element $(\vec{v}, \rho)$. In wreath product $C_m \wr S_n$, there is a 1-1 correspondence between the conjugacy classes and the type matrices.

Proposition 3 (Conjugacy Classes of $C_m \wr S_n$). *1. $(\vec{v}, \rho) \sim (\vec{v}', \rho')$ in $C_m \wr S_n$ iff they have same type-matrix, i.e., $a(\vec{v}, \rho) = a(\vec{v}', \rho')$.*

2. The number of conjugacy classes in $C_m \wr S_n$ is equal to

$$\sum_{(n_1,\ldots,n_m)} p(n_1) \cdot \cdots \cdot p(n_m), \tag{2}$$

where the sum is over all m-tuple of non-negative integers such that $\sum_{i=1}^m n_i = n$ and $p(n_i)$ is the number of partitions of integer n_i (with $p(0) := 1$).

3. The size of the conjugacy class with type-matrix $a(\vec{v}, \rho)$ is

$$\frac{m^n \cdot n!}{\prod_{i,j}[(jm)^{a_{ij}} \cdot a_{ij}!]} \tag{3}$$

and the size of the centralizer of $(\vec{v}, \rho)$ is

$$\prod_{i,j}[(jm)^{a_{ij}} \cdot a_{ij}!]. \tag{4}$$

Proof. This is Theorem 4.2.8 (p141), Lemma 4.2.9 (p142), and Lemma 4.2.10 (p143) of James and Kerber (1981) [6]. □

Looking closely at the proof of the 'if' part of 1, we get a sufficent condition for a conjugacy class in $C_m \wr S_n$ *not* split in the subgroup $C_m^{n-1} \rtimes S_n$.

Lemma 4. *Let $a(\vec{v}, \sigma)$ be a type-matrix with $\sum v_i = 0$. If the cycle decomposition of σ contains a cycle of length k such that $(m, k) = 1$, then the conjugacy class in $C_m \wr S_n$ assocated to $a(\vec{v}, \sigma)$ does not split in subgroup $C_m^{n-1} \rtimes S_n$. In particular, if σ fixes any points (contains cycles of length 1), then the conjugacy class does not split.*

Proof. Suppose both $(\vec{v}, \sigma)$ and $(\vec{v}', \sigma')$ have same type-matrix. Then we can find a $\rho \in S_n$ (may not be unique) that maps cycles in σ to corresponding cylces in σ' with same length and cycle product. Specifically, for a cycle, say $(c'_1 \dots c'_k)$, in σ', we choose a cycle (not chosen before), say $(c_1 \cdots c_k)$, in σ which has the same length k and same cycle product and set $\rho(c'_i) = c_i, 1 \le i \le k$. Repeat this for all cycles in σ' we get a $\rho \in S_n$. ρ thus defined satisfies $\sigma = \rho\sigma'\rho^{-1}$ and hence $(\vec{v}'_\rho, \sigma) = (\vec{e}, \rho)(\vec{v}', \sigma')(\vec{e}, \rho)^{-1}$, i.e., $(\vec{v}'_\rho, \sigma)$ is conjugate with $(\vec{v}', \sigma')$. So it is enough to show $(\vec{v}'_\rho, \sigma)$ is conjugate with $(\vec{v}, \sigma)$. We construct a $\vec{w} \in C_p^n$ such that $(\vec{w}, 1)(\vec{v}'_\rho, \sigma)(\vec{w}^{-1}, 1) = (\vec{w}\vec{v}'_\rho\vec{w}_\sigma^{-1}, \sigma) = (\vec{v}, \sigma)$. Fix any cycle, say $(c_1 \cdots c_k)$, in σ, note its cycle product wrt $\vec{v}$ or $\vec{v}'_\rho$ are equal. Set w_{c_1} to be any element in C_p. We need $w_{c_1}(\vec{v}'_\rho)_{c_1}\vec{w}^{-1}_{\sigma^{-1}(c_1)} = (\vec{v})_{c_1}$, i.e., $w_{c_1}v'_{\rho^{-1}(c_1)}w^{-1}_{c_k} = v_{c_1}$. So we can solve to get w_{c_k} and in turn $w_{c_{k-1}}, \dots, w_{c_2}$. Repeat this procedure for all cycles in σ. We obtain the desired $\vec{w}$. But we require $\vec{w}$ to satisfy $\prod_i w_i = 0$. Note that the recursion $w_{c_r}(\vec{v}'_\rho)_{c_r}\vec{w}^{-1}_{\sigma^{-1}(c_r)} = (\vec{v})_{c_r}$ implies that the cycle product of $\vec{w}$ is $w_{c_1}^k \cdot$ constant. We have total freedom in choosing w_{c_1}. Thus if $(k, m) = 1$, this cycle product can take any value in C_m and thus make the total product of $\vec{w}$ to be 0. □

Corollary 5. *Assume p prime and $(p, n) = 1$. The conjugacy classes in $C_p \wr S_n$ with total vector product 0 are also conjugacy classes in the subgroup $C_p^{n-1} \rtimes S_n$. They do not split and constitute all conjugacy classes in $C_p^{n-1} \rtimes S_n$.*

Proof. There is no way for a permutation in S_n to have cycles of lengths only multiples of p. □

Example 6. All conjugacy classes in $C_3 \wr S_8$ either lie entirely outside $C_3^7 \rtimes S_8$ (total twists is not 0) or lie entirely inside $C_3^7 \rtimes S_8$ (total twists 0) and do not split. The number of conjugacy classes is 810 in $C_3 \wr S_8$ and 270 in $C_3^7 \rtimes S_8$. (Why this is true? Justify!) Let $\pi_i, 1 \le i \le 3$, be the partition corresponding ith row of type matrix. Then classes in $C_3^7 \rtimes S_8$ can be labelled by 3-tuples of partition of n, $\{(\pi_1, \pi_2, \pi_3) | \sum_{i=1}^3 |\pi_i| = 8, \sum_{i,j} j\pi_{ij} = 0\}$.

Example 7. The 12 quarter turns of the $2 \times 2 \times 2$ Rubik's cube all have type matrix

	1	2	3	4	5	6	7	8
0	4			1				
1								
2								

and thus are conjugate to each other in $C_3 \wr S_8$. This conjugacy class has size $\frac{3^8 \times 8!}{(3^4 \cdot 4!) \times (12)} = 11340$ and does not split in $C_3^7 \rtimes S_8$.

Example 8. The moves "switch two subcubes of a $2 \times 2 \times 2$ Rubik's cube and twist one of them by $c \in C_3$ and the other one by c^{-1}" constitute a conjugacy class in $C_3 \wr S_8$ with type matrix

	1	2	3	4	5	6	7	8
0	6	1						
1								
2								

This conjugacy class has size $\frac{3^8 \times 8!}{(3^6 \cdot 6!) \times (6)} = 84$ and does not split in subgroup $C_3^7 \rtimes S_8$.

Example 9 (Counter Example). Some conjugacy classes in $C_2 \wr S_{12}$ split but some do not. See Lemma 23.

Question 10. *For general m, n, what are the sufficient and necessary conditions for the a conjugacy classe in $C_m \wr S_n$ to split in subgroup $C_m^{n-1} \rtimes S_n$?*

Next step is to figure out

Question 11. *How do the irreducible characters of $C_3 \wr S_8$ split in the subgroup $C_3^7 \rtimes S_8$?*

Because $C_3^7 \rtimes S_8$ is a normal subgroup of index 3 in $C_3 \wr S_8$, it is natural to apply Clifford theorem.

Proposition 12 (Clifford theorem). *Suppose that $H \lhd G$ and χ is an irreducible character of G. Then*
(1) all the constituents of $\chi \downarrow H$ have the same degree; and
(2) if $\psi_1, \dots, \psi_m$ are the constituents of $\chi \downarrow H$, then

$$\chi \downarrow H = e(\psi_1 + \cdots + \psi_m)$$

where e is an non-negative integer satisfying $me^2 \leq |G : H|$.

Proof. This is Proposition 20.5 (p214) and Theorem 20.8 (p216) in James and Kerber (2001) [7]. □

Applying Clifford theorem to normal subgroup of index 3, we have

Proposition 13. *Suppose that H is a normal subgroup of index 3 in G and let χ be an irreducible character of G. Exactly one of following three cases happens.*
(1) $\chi \downarrow H$ is irreducible, or
(2) $\chi \downarrow H = \psi_1 + \psi_2$ is sum of two irreducible characters of H of same degree, or
(3) $\chi \downarrow H = \psi_1 + \psi_2 + \psi_3$ is sum of three irreducible characters of H of same degree.

Proof. By Proposition 12, $e = 1$. So $m = 1, 2$ or 3. □

We lift the two non-trivial linear characters of $G/H = C_3$ to obtain two linear characters of G. Let $\{H_0 = H, H_1, H_2\}$ be cosets of H in G and $\omega = e^{2\pi\sqrt{-1}/3}$. Then

$$\lambda(g) = \begin{cases} 1 & g \in H_0 \\ \omega & g \in H_1 \\ \omega^2 & g \in H_2 \end{cases} \text{ and } \lambda^2(g) = \begin{cases} 1 & g \in H_0 \\ \omega^2 & g \in H_1 \\ \omega & g \in H_2 \end{cases}, g \in G,$$

are two linear characters of G. For any irreducible character χ of G, $\chi, \lambda\chi$ and $\lambda^2\chi$ are three irreducible characters of same degree (c.f. Proposition 17.14 of James and Kerber (2001) [7] p176). Also $\chi \downarrow H = \lambda\chi \downarrow H = \lambda^2\chi \downarrow H$ because $\lambda(h) = \lambda^2(h) = 1$ for all $h \in H$.

First we study case (3) in Proposition 13.

Proposition 14. *Suppose that H is a normal subgroup of index 3 in G, and that χ is an irreducible character of G. Then the following three conditions are equivalent:*
(1) $\chi \downarrow H = \psi_1 + \psi_2 + \psi_3$ is a sum of three (pairwise inequivalent) irreducible characters of H;
(2) $\chi(g) = 0$ for all $g \notin H$;
(3) $\chi = \lambda\chi = \lambda^2\chi$.

Proof. Note

$$\lambda\chi(g) = \begin{cases} \chi(g) & g \in H_0 \\ \omega\chi(g) & g \in H_1 \\ \omega^2\chi(g) & g \in H_2 \end{cases} \text{ and } \lambda^2\chi(g) = \begin{cases} \chi(g) & g \in H_0 \\ \omega^2\chi(g) & g \in H_1 \\ \omega^2\chi(g) & g \in H_2 \end{cases}, g \in G,$$

We know the equility holds in $me^2 \leq 3$ (i.e., $m = 3$) iff $\chi(g) = 0$ for all $g \notin H$ (c.f. Proposition 20.5 in James and Kerber (2001) [7] p214). Thus (1) is equivalent to (2). Equivalence of (2) and (3) is obvious. □

Proposition 15. *Suppose that H is a normal subgroup of index 3 in G, and that χ is an irreducible characters of G such that $\chi \downarrow H = \psi_1 + \psi_2 + \psi_3$ is a sum of three irreducible characters of H. If ϕ is any irreducible character of G such that $\phi \downarrow H$ has ψ_1 or ψ_2 or ψ_3 as a constituent, then $\phi = \chi$.*

Proof. By Proposition 14, $\chi = \lambda\chi = \lambda^2\chi$ and $\chi(g) = 0$ for all $g \notin H$. Thus

$$< \phi, \chi >_G = \frac{1}{|G|}\sum_{g\in H} \phi(g)\bar{\chi}(g) = \frac{1}{3} < \phi \downarrow H, \chi \downarrow H >_H .$$

ϕ is irreducible by hypothesis thus $< \phi, \chi >_G = 0$ or 1. If $\phi \downarrow H$ contains any ψ_i as a constituent, then RHS $\neq 0$ and hence $\phi = \chi$. □

Next we study case (1) in Proposition 13.

Proposition 16. *Suppose that H is a normal subgroup of index 3 in G, and that χ is an irreducible characters of G such that $\chi \downarrow H$ is irreducible. If ϕ is any character of G such that $\phi \downarrow H = \chi \downarrow H$, then $\phi = \chi, \lambda\chi$, or $\lambda^2\chi$.*

Proof. Note that

$$(\chi + \lambda\chi + \lambda^2\chi)(g) = \begin{cases} 3\chi(g) & g \in H_0 = H \\ 0 & g \notin H \end{cases}$$

So

$$\begin{aligned} < \chi + \lambda\chi + \lambda^2\chi, \phi >_G &= \frac{1}{|G|}\sum_{g\in H} 3\chi(g)\bar{\phi}(g) = \frac{1}{|H|}\sum_{g\in H} \chi(g)\bar{\phi}(g) \\ &= < \chi \downarrow H, \phi \downarrow H >_H = < \chi \downarrow H, \chi \downarrow H >_H = 1 \end{aligned}$$

and the result follows. □

Finally we show that case (2) of Proposition 13 can not happen!

Proposition 17. *Suppose that H is a normal subgroup of index 3 in G, and that χ is an irreducible characters of G. Then $\chi \downarrow H$ can not be a sum of two irreducible characters of H.*

Proof. Suppose $\chi \downarrow H = \psi_1 + \psi_2$ where ψ_is are irreducible characters of H. Then for any character ϕ of G such that $\phi \downarrow H = \chi \downarrow H$, we have

$$< \chi + \lambda\chi + \lambda^2\chi, \phi >_G = < \chi \downarrow H, \phi \downarrow H >_H = < \chi \downarrow H, \chi \downarrow H >_H = 2.$$

Now take $\phi = \chi$ or $\lambda\chi$ or $\lambda^2\chi$, LHS = 1, a contradiction. □

In summary, let H be a normal subgroup of index 3 in group G, then
(1) Each irreducible character χ of G which is nonzero somewhere outside H restricts to be an irreducible character H. Such characters of G occur in triplets ($\chi, \lambda\chi$ and $\lambda^2\chi$) which have the same restriction to H.
(2) If χ is an irreducible character of G which is zero everywhere outside H, then χ restricts to be the sum of three distinct irreducible characters of H of the same degree. The three characters of H which we get from χ in this way come from no other irreducible character of G.
(3) Every irreducible character of H appears among those obtained by restricting irreducible characters of G, as in parts (1) and (2).

Now in our case, $C_3^7 \rtimes S_8$ is a subgroup of index 3 in $C_3 \wr S_8$. It is the kernel of group homomorphism $\varphi : C_3 \wr S_8 \to C_3, (\vec{w}, \rho) \mapsto \sum_{i=1}^{3} w_i$, and thus is normal. Now we know number of classes/irreducible characters in $C_3 \wr S_8$ is 810 and that of $C_3^7 \rtimes S_8$ is 270. Thus only case (1) happens, i.e., no irreducible character of $C_3 \wr S_8$ splits in $C_3^7 \rtimes S_8$. Therefore

Morever, $\lambda\chi^{(\alpha_1,\alpha_2,\alpha_3)} = \chi^{(\alpha_3,\alpha_1,\alpha_2)}$ and $\lambda^2\chi^{(\alpha_1,\alpha_2,\alpha_3)} = \chi^{(\alpha_2,\alpha_3,\alpha_1)}$. Therefore $\chi^{(\alpha_1,\alpha_2,\alpha_3)}$, $\chi^{(\alpha_3,\alpha_1,\alpha_2)}$ and $\chi^{(\alpha_2,\alpha_3,\alpha_1)}$ are the self-associated representations and restrict to the same irreducible representation of $C_3 \rtimes S_8$. This also shows that $\chi^{(\alpha_1,\alpha_2,\alpha_3)}$ will split iff $\alpha_1 = \alpha_2 = \alpha_3$. This verifies that the number of conjugacy classes in $C_3 \rtimes S_8$ (270) should be 1/3 of that of $C_3 \wr S_8$ (810).

Corollary 18. *The character table of $C_3^7 \rtimes S_8$ is a subtable (270×270) of the character table (810×810) of $C_3 \wr S_8$.*

Question 19. *For general m, n, how do the irreducible characters of $C_m \wr S_n$ split in subgroup $C_m^{n-1} \rtimes S_n$?*

For these questions, we may get hints from the proof for Weyl group of type D_n or from the computer analysis ...

3.2 Computer analysis of G_V

With aid of computer, we are able to do eigen-analysis of certain random walks driven by conjugacy classes in G_V. All results are obtained using GAP 4.4 [5] and Matlab.

GAP computes the character table of $G_V \cong C_3^7 \rtimes S_8$, which gives much information about this group. We compile some of the information here

- G_V has 270 conjugacy classes and irreducible characters.
- The degrees of irreducible characters are

```
[ [ 1, 2 ], [ 7, 2 ], [ 8, 4 ], [ 14, 2 ], [ 20, 2 ], [ 21, 2 ], [ 28, 10 ],
  [ 35, 2 ], [ 42, 1 ], [ 48, 4 ], [ 56, 12 ], [ 64, 2 ], [ 70, 6 ],
  [ 90, 1 ], [ 112, 12 ], [ 120, 4 ], [ 140, 20 ], [ 160, 2 ], [ 168, 12 ],
  [ 210, 8 ], [ 224, 8 ], [ 252, 8 ], [ 280, 33 ], [ 336, 4 ], [ 420, 12 ],
  [ 448, 8 ], [ 504, 2 ], [ 560, 22 ], [ 630, 4 ], [ 672, 10 ], [ 840, 20 ],
  [ 896, 1 ], [ 1008, 4 ], [ 1120, 10 ], [ 1260, 8 ], [ 1680, 4 ],
  [ 2240, 2 ] ]
```

  Here a pair $[d_\rho, n_{d_\rho}]$ means that there are n_{d_ρ} characters of degree d_ρ.

- Sizes of conjugacy classes, by formula (3), are

```
[ 1, 56, 56, 56, 420, 280, 280, 560, 28, 28, 168, 168, 70, 8, 8, 272160,
  544320, 544320, 272160, 544320, 272160, 181440, 362880, 181440, 90720,
  362880, 181440, 181440, 90720, 362880, 181440, 90720, 181440, 5040, 504,
  1260, 1260, 5040, 2520, 1680, 84, 1680, 7560, 5040, 1260, 504, 5040, 1260,
  5040, 2520, 2520, 504, 5040, 504, 504, 1260, 84, 1680, 1260, 504, 84,
  136080, 136080, 136080, 45360, 68040, 45360, 45360, 68040, 11340, 45360,
  68040, 45360, 11340, 45360, 11340, 7560, 22680, 45360, 15120, 11340, 22680,
  22680, 15120, 45360, 7560, 45360, 22680, 1890, 15120, 3780, 7560, 22680,
  11340, 7560, 15120, 15120, 3780, 7560, 1890, 11340, 22680, 15120, 7560,
  3780, 1890, 816480, 816480, 816480, 816480, 816480, 816480, 816480, 816480,
  816480, 272160, 136080, 272160, 136080, 272160, 136080, 272160, 136080,
  272160, 136080, 272160, 136080, 272160, 136080, 272160, 136080, 272160,
  136080, 5040, 20160, 10080, 1008, 5040, 10080, 30240, 10080, 30240, 10080,
  30240, 20160, 20160, 5040, 5040, 10080, 5040, 10080, 5040, 1008, 1008,
  408240, 204120, 408240, 408240, 204120, 204120, 408240, 204120, 408240,
  408240, 204120, 204120, 408240, 204120, 408240, 408240, 204120, 204120,
  979776, 979776, 979776, 979776, 979776, 979776, 979776, 979776, 979776,
  326592, 326592, 653184, 108864, 326592, 326592, 108864, 326592, 326592,
  108864, 1959552, 1959552, 1959552, 11022480, 1837080, 918540, 102060,
  34020, 34020, 8505, 51030, 306180, 612360, 612360, 306180, 612360, 306180,
  136080, 22680, 22680, 22680, 34020, 34020, 34020, 68040, 68040, 68040,
  68040, 68040, 68040, 68040, 11340, 11340, 11340, 34020, 34020, 34020,
  816480, 1632960, 1632960, 1632960, 816480, 816480, 2449440, 2449440,
  2449440, 30240, 30240, 30240, 90720, 90720, 90720, 90720, 90720, 90720,
  181440, 181440, 181440, 90720, 90720, 90720, 90720, 90720, 90720, 90720,
  90720, 90720, 90720, 90720, 90720, 30240, 30240, 30240, 30240, 30240,
  30240, 4199040, 4199040, 4199040 ]
```

- Sizes of centralizers of each conjugacy class (in same order), by formula (4), are

```
[ 88179840, 1574640, 1574640, 1574640, 209952, 314928, 314928, 157464,
  3149280, 3149280, 524880, 524880, 1259712, 11022480, 11022480, 324, 162,
  162, 324, 162, 324, 486, 243, 486, 972, 243, 486, 486, 972, 243, 486, 972,
  486, 17496, 174960, 69984, 69984, 17496, 34992, 52488, 1049760, 52488,
  11664, 17496, 69984, 174960, 17496, 69984, 17496, 34992, 34992, 174960,
  17496, 174960, 174960, 69984, 1049760, 52488, 69984, 174960, 1049760, 648,
  648, 648, 1944, 1296, 1944, 1944, 1296, 7776, 1944, 1296, 1944, 7776,
  1944, 7776, 11664, 3888, 1944, 5832, 7776, 3888, 3888, 5832, 1944, 11664,
  1944, 3888, 46656, 5832, 23328, 11664, 3888, 7776, 11664, 5832, 5832,
  23328, 11664, 46656, 7776, 3888, 5832, 11664, 23328, 46656, 108, 108, 108,
  108, 108, 108, 108, 108, 108, 324, 648, 324, 648, 324, 648, 324, 648, 324,
  648, 324, 648, 324, 648, 324, 648, 324, 648, 17496, 4374, 8748, 87480,
  17496, 8748, 2916, 8748, 2916, 8748, 2916, 4374, 4374, 17496, 17496, 8748,
  17496, 8748, 17496, 87480, 87480, 216, 432, 216, 216, 432, 432, 216, 432,
  216, 216, 432, 432, 216, 432, 216, 216, 432, 432, 90, 90, 90, 90, 90, 90,
```

```
90, 90, 90, 270, 270, 135, 810, 270, 270, 810, 270, 270, 810, 45, 45, 45,
8, 48, 96, 864, 2592, 2592, 10368, 1728, 288, 144, 144, 288, 144, 288,
648, 3888, 3888, 3888, 2592, 2592, 2592, 1296, 1296, 1296, 1296, 1296,
1296, 1296, 7776, 7776, 7776, 2592, 2592, 2592, 108, 54, 54, 54, 108, 108,
36, 36, 36, 2916, 2916, 2916, 972, 972, 972, 972, 972, 972, 486, 486, 486,
972, 972, 972, 972, 972, 972, 972, 972, 972, 972, 972, 972, 2916, 2916,
2916, 2916, 2916, 2916, 21, 21, 21 ]
```

- Orders of the representatives of each conjugacy class (in same order) are

```
[ 1, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 18, 18, 18, 6, 18, 18, 9, 9,
  9, 3, 9, 9, 9, 9, 9, 3, 9, 9, 6, 6, 6, 6, 6, 6, 6, 2, 6, 6, 6, 6, 6, 6, 6,
  6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 12, 12, 12, 12, 12, 12, 12, 12, 4,
  12, 12, 12, 12, 12, 12, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 2, 6, 6, 6, 6,
  6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 36, 12, 36, 12, 36, 36, 36, 36, 12,
  18, 6, 18, 18, 6, 18, 6, 18, 18, 18, 18, 6, 18, 18, 6, 6, 18, 18, 9, 3, 9,
  3, 9, 9, 9, 3, 9, 3, 3, 9, 9, 9, 9, 9, 3, 9, 3, 9, 9, 12, 12, 12, 12, 12,
  12, 12, 4, 12, 12, 12, 12, 12, 12, 12, 12, 12, 12, 30, 30, 30, 30, 30, 30,
  10, 30, 30, 15, 15, 15, 5, 15, 15, 15, 15, 15, 15, 45, 15, 45, 8, 12, 4,
  6, 6, 6, 2, 6, 12, 12, 12, 4, 12, 12, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6,
  6, 6, 6, 6, 2, 6, 6, 6, 18, 18, 6, 18, 18, 6, 18, 6, 18, 18, 18, 6, 6, 18,
  18, 18, 6, 18, 18, 18, 6, 6, 18, 18, 18, 6, 18, 6, 18, 18, 18, 6, 18, 18,
  18, 6, 18, 18, 6, 7, 21, 21 ]
```

3.3 Random walk driven by the "Constricted Paired Twists"

This is slight variation of the "paired shuffle" considered by Schoolfield (2002) [13] except that we have to conserve total twists. We can think of this as the random transposition on the semi-direct product $C_m^{n-1} \rtimes S_n, n > 2$. Consider a line of n C_m-wheels. At each step, we pick two random integers i and j from $\{1, 2, \ldots, n\}$ independently. If $i = j$, we do nothing. If $i \neq j$, we switch the ith and jth wheels, rotate the new ith wheel by $c \in C_m$ uniformly and the new jth wheel by c^{-1}. The probability measure is

$$P(\vec{v}, \sigma) = \begin{cases} \frac{1}{n} & \text{if } \vec{v} = \vec{e}, \sigma = 1 \\ \frac{2}{mn^2} & \text{if } \vec{v} = \vec{e} \text{ or has two non-idendity entries}, \sigma \text{ is a transposition} \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

By Lemma 4, P is a class function with support on the idendity and the conjugacy class of size $\frac{m^n \cdot n!}{(m^{n-2}(n-1)!) \cdot (2m)} = \frac{mn(n-1)}{2}$. This conjugacy class is symmetric since inverse of each element in this class has the same type-matrix. Now, e.g., Corollary 3 of Diaconis and Shahshahani (1981) [2] says

Proposition 20. *Let P be a probability measure on a finite group G and P is a class function. Let $\mathbf{P}$ be the transition matrix of the Markov chain induced by the probability measure P. Then, for each irreducible representation ρ (of degree d_ρ) of G, there is an eigenvalue π_ρ of $\mathbf{P}$ with algebraic multiplicity d_ρ^2 such that*

$$\pi_\rho = \frac{1}{d_\rho} < P, \chi_\rho >_G := \frac{1}{d_\rho} \sum_{i=1}^{s} P_i n_i \chi_i, \tag{6}$$

where P_i is the value of P on ith conjugacy class, n_i is the size of the ith conjugacy class, χ_i is the character value of ρ at the ith conjugacy class, and the sum is over all conjugacy classes.

Numerically, the eigenvalues (sorted), corresponding to each irreducible representation, for the "constricted paired twists" random walk on the $2 \times 2 \times 2$ Rubik's cube group G_V are

```
Columns 1 through 10
 -0.7500   -0.5313   -0.5313   -0.5000   -0.3750   -0.3750   -0.3438   -0.3125   -0.3125   -0.3125
Columns 11 through 20
 -0.3125   -0.3125   -0.2813   -0.2813   -0.2500   -0.2500   -0.2188   -0.2188   -0.1875   -0.1875
Columns 21 through 30
 -0.1875   -0.1875   -0.1875   -0.1563   -0.1563   -0.1563   -0.1563   -0.1563   -0.1563   -0.1563
Columns 31 through 40
 -0.1250   -0.1250   -0.1250   -0.1250   -0.1250   -0.1250   -0.1250   -0.1250   -0.0938   -0.0938
Columns 41 through 50
 -0.0938   -0.0938   -0.0938   -0.0938   -0.0625   -0.0625   -0.0625   -0.0625   -0.0625   -0.0625
Columns 51 through 60
 -0.0625   -0.0625   -0.0625   -0.0625   -0.0625   -0.0625   -0.0313   -0.0313   -0.0313   -0.0313
Columns 61 through 70
 -0.0313   -0.0313   -0.0313   -0.0313         0         0         0         0         0         0
Columns 71 through 80
       0         0         0         0         0         0         0         0         0         0
Columns 81 through 90
       0    0.0313    0.0313    0.0313    0.0313    0.0313    0.0313    0.0313    0.0313    0.0313
Columns 91 through 100
  0.0313    0.0313    0.0313    0.0625    0.0625    0.0625    0.0625    0.0625    0.0625    0.0625
Columns 101 through 110
  0.0625    0.0625    0.0625    0.0625    0.0625    0.0625    0.0625    0.0625    0.0625    0.0625
Columns 111 through 120
  0.0625    0.0938    0.0938    0.0938    0.0938    0.0938    0.0938    0.0938    0.0938    0.0938
Columns 121 through 130
  0.0938    0.0938    0.0938    0.0938    0.0938    0.0938    0.1250    0.1250    0.1250    0.1250
Columns 131 through 140
  0.1250    0.1250    0.1250    0.1250    0.1250    0.1250    0.1250    0.1250    0.1250    0.1250
Columns 141 through 150
  0.1250    0.1250    0.1250    0.1250    0.1563    0.1563    0.1563    0.1563    0.1563    0.1563
Columns 151 through 160
  0.1563    0.1563    0.1563    0.1563    0.1563    0.1563    0.1563    0.1563    0.1563    0.1875
Columns 161 through 170
  0.1875    0.1875    0.1875    0.1875    0.1875    0.1875    0.1875    0.1875    0.1875    0.1875
Columns 171 through 180
  0.1875    0.1875    0.1875    0.1875    0.1875    0.1875    0.1875    0.2188    0.2188    0.2188
Columns 181 through 190
  0.2188    0.2188    0.2188    0.2188    0.2188    0.2188    0.2188    0.2188    0.2188    0.2500
Columns 191 through 200
  0.2500    0.2500    0.2500    0.2500    0.2500    0.2500    0.2500    0.2500    0.2500    0.2500
Columns 201 through 210
  0.2500    0.2500    0.2500    0.2500    0.2500    0.2500    0.2813    0.2813    0.2813    0.2813
Columns 211 through 220
  0.2813    0.2813    0.2813    0.2813    0.3125    0.3125    0.3125    0.3125    0.3125    0.3125
Columns 221 through 230
  0.3125    0.3125    0.3125    0.3125    0.3125    0.3125    0.3438    0.3438    0.3438    0.3438
Columns 231 through 240
  0.3438    0.3438    0.3750    0.3750    0.3750    0.3750    0.3750    0.3750    0.3750    0.3750
Columns 241 through 250
  0.4063    0.4063    0.4063    0.4063    0.4063    0.4063    0.4063    0.4375    0.4375    0.4375
Columns 251 through 260
  0.4375    0.4375    0.4688    0.4688    0.5000    0.5000    0.5313    0.5313    0.5625    0.5625
Columns 261 through 270
  0.5625    0.5625    0.5625    0.5938    0.6250    0.6250    0.7500    0.7813    0.7813    1.0000
```

By the Upper Bound Lemma in Diaconis (1988) [1], we have

Proposition 21. *Let P be a symmetric probability measure on a finite group G and $1 = \beta_0 \geq \beta_1 \geq \ldots \geq \beta_{|G|-1} \geq -1$ be the associated eigenvalues, then*

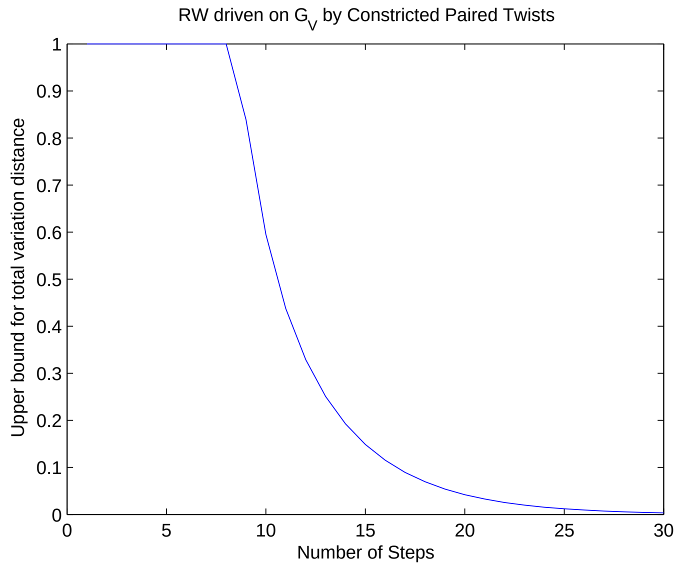
$$\|P^{(n)} - U\|_{TV} \leq \frac{1}{2}|G|^{1/2}\|P^{(n)} - U\|_2 = \frac{1}{2}\left(\sum_{i=1}^{|G|-1} \beta_i^{2n}\right)^{1/2} \leq \frac{1}{2}(|G|-1)^{1/2}\beta^{*n}. \tag{7}$$

In case P is a class function, then we have

$$\|P^{(n)} - U\|_{TV} \leq \frac{1}{2}|G|^{1/2}\|P^{(n)} - U\|_2 = \frac{1}{2}\left(\sum_{\rho} d_\rho^2 \pi_\rho^{2n}\right)^{1/2}, \tag{8}$$

where the sum is over all non-trivial irreducible representations of G, d_ρ and π_ρ are defined as in last proposition.

Below is the graph of the upper bound for total variation of the random walk driven by the "constricted paried twists" on the $2 \times 2 \times 2$ Rubik's cube group G_V. Around twenty steps are enough to bring the total variation low.



4 Edge Subcubes of $3 \times 3 \times 3$ Rubik's Cube

Let G_E denote the group of all (legal) moves of the 12 edge subcubes of a $3 \times 3 \times 3$ Rubik's cube. Then G_E is a subgroup of $H_E \cong C_2 \wr S_{12}$ of index 2, i.e., $G_E \cong C_2^{11} \rtimes S_{12}$ contains all moves in H_E that conserve the total flips. $|G_E| = 2^{12}12!/2 = 980,995,276,800 \approx 9.8 \times 10^{11}$. We have relatively more knowledge of this group because it is a Weyl group of type D_{12}.

Question 22. *Any good references for conjugacy classes and irreducible characters of Weyl group of type D_n?*

4.1 Conjugacy classes and irreducible characters of G_E

Lemma 23. *Conjugacy class $a(\vec{v}, \rho)$ in $C_2 \wr S_n$ splits into two conjugacy classes of equal size in $C_2^{n-1} \rtimes S_n$ iff all cycles have even length and cycle product 0. In particular, when n is odd, no conjugacy classes in $C_2 \wr S_n$ split in $C_2^{n-1} \rtimes S_n$.*

Proof. Case 1: If any cycle has odd length, then the conjugacy class does not split in $C_2^{n-1} \rtimes S_n$ by Lemma 4. Case 2: Suppose all cycles have even length and at least one has cycle product 1. We work harder on the proof for Lemma 4. For a cycle of length k (even) and cycle product 1 wrt $\vec{v}$, recursion $w_{c_r} v_{\rho^{-1}(c_r)} w^{-1}_{\sigma^{-1}(c_r)} = v_{c_r}$ implies that the cycle product wrt $\vec{w}$ is

$$\begin{aligned} w_{c_1}\cdots w_{c_r} &= w_{c_1}^k\left(\frac{v_{c2}}{v'_{c2}}\right)\left(\frac{v_{c2}v_{c3}}{v'_{c2}v'_{c3}}\right)\cdots\left(\frac{v_{c2}\cdots v_{c_k}}{v'_{c2}\cdots v'_{c_k}}\right)=\left(\frac{v_{c2}}{v'_{c2}}\right)\left(\frac{v_{c2}v_{c3}}{v'_{c2}v'_{c3}}\right)\cdots\left(\frac{v_{c2}\cdots v_{c_k}}{v'_{c2}\cdots v'_{c_k}}\right) \\ &= \left(\frac{v_{c2}}{v'_{c2}}\right)^{k-1}\left(\frac{v_{c3}}{v'_{c3}}\right)^{k-2}\cdots\left(\frac{v_{k-1}}{v'_{k-1}}\right)^{2}\left(\frac{v_{c_k}}{v'_{c_k}}\right)=\left(\frac{v_{c2}}{v'_{c2}}\right)\cdots\left(\frac{v_{c_k}}{v'_{c_k}}\right)=\frac{v_{c2}v_{c4}\cdots v_{c_k}}{v'_{c2}v'_{c4}\cdots v'_{c_k}}. \end{aligned}$$

Now since $\prod_{i=1}^k v_{c_i}=1$, $v_{c_1}v_{c_3}\cdots v_{c_{k-1}}$ and $v_{c_2}v_{c4}\cdots v_{c_k}$ take different values in $\{0,1\}$. So by varying ρ that maps the cycle in $\vec{v}$ and $\vec{v}'$. We can always make the total product of $\vec{w}$ to be 0. Hence the conjugacy class does not split in this case. Case 3: Now suppose all cycles have even length and cycle product 0. Since $\prod_{i=1}^k v_{c_i}=0$, $v_{c_1}v_{c_3}\cdots v_{c_{k-1}}=v_{c2}v_{c4}\cdots v_{c_k}$ in this case. For any permutation ρ, the total product of $\vec{w}$ is $\prod_{c \text{ cycle}}\frac{v_{c_2}v_{c_4}\cdots v_{c_k}}{v'_{c_2}v'_{c_4}\cdots v'_{c_k}}$. The factors in denominator and numerator will cancel out because of last equality. □

Question 24. *Generalize the idea/proof for conjugacy classes in $C_m^{n-1}\rtimes S_n$?*

If the classes of $C_2\wr S_{12}$ are labelled by pairs of partitions (π_1,π_2) where π_i is the partition corresponding to ith row of the type matrix, then a class (π_1,π_2) in $C_2\wr S_{12}$ will split in subgroup $C_2^{11}\rtimes S_{12}$ iff π_2 is empty and π_1 consists of only even parts (i.e., $\pi_1=2\pi$ for some partition π of $n/2$). So the index set of classes in $C_2^{11}\rtimes S_{12}$ is

$$\{(\pi_1,\pi_2)|\pi_2 \text{ is non-empty or } \pi_1 \text{ contains odd parts}\}\cup\{(2\pi,[])_\pm\}.$$

Example 25. The 12 quarter turns in G_E all have type matrix

	1	2	3	4	5	6	7	8	9	11	12
0	8			1							
1											

and thus are conjugate to each other in $C_2\wr S_{12}$. This conjugacy class contains 1-cycles and thus does not split in $C_2^{11}\rtimes S_{12}$. It has size $\frac{2^{12}\times 12!}{(2^8\cdot 8!)\times(8)}=23760$.

Example 26. The moves "switch two edge subcubes and flip one of them by $c\in C_2$ and the other by c^{-1}" constitute a conjugacy class in $C_2\wr S_{12}$ with type matrix

	1	2	3	4	5	6	7	8	9	10	11	12
0	10	1										
1												

This conjugacy class has size $\frac{2^{12}\times 12!}{(2^{10}\cdot 10!)\times(4)}=132$ and does not split in subgroup $C_2^{11}\rtimes S_{12}$.

Lemma 27. *Let the irreducible characters of $C_2\wr S_n$ be indexed by pair of partitions (α_1,α_2) where α_i is the partition corresponding to ith row of the type matrix. The characters $\chi^{(\alpha_1,\alpha_2)}$ and $\chi^{(\alpha_2,\alpha_1)}$ of $C_2\wr S_n$ will restrict to the same irreducible characters of $C_2^{n-1}\rtimes S_n$ for $\alpha_1\neq\alpha_2$. The characters $\chi^{(\alpha,\alpha)}$ of $C_2\wr S_n$ will restrict to a sum of two irreducible characters $\chi_+^{(\alpha,\alpha)}$ and $\chi_-^{(\alpha,\alpha)}$ of $C_2^{n-1}\rtimes S_n$. In particular, when n is odd, no irreducible characters of $C_2\wr S_n$ will split in subgroup $C_2^{n-1}\rtimes S_n$.*

Proof. Any good references??? □

Question 28. *Generalize the idea/proof for irreducible characters of $C_m^{n-1} \rtimes S_n$?*

What happens to the split characters on the split classes?

Lemma 29. *Let $n > 0$ be an even integer and α, π be partitions of $n/2$. Then the character value $\chi_{\pm}^{(\alpha,\alpha)}$ on a class $(2\pi, [])_{\pm}$ is determined by*

$$\chi_{\epsilon}^{(\alpha,\alpha)}((2\pi, [])_{\delta}) = \frac{1}{2}\chi^{(\alpha,\alpha)}(2\pi, []) + \epsilon\delta 2^{l(\pi)-1}\chi^{\alpha}(\pi).$$

Proof. This is Theorem 5.1 in Pfeiffer (1994) [10] □

4.2 Computer analysis of G_E

GAP computes the character table of $G_E \cong C_2^{11} \rtimes S_{12}$:

- G_E has 599 conjugacy classes and irreducible characters; The character table is rational integral (justify? James and Kerber p161 Ex 4.4) Actually both C_2 and S_n are ambivalent, so $C_2 \wr S_n$ is ambivalent (Lemma 4.2.13 p144 James and Kerber (1981) [6]).
- The degrees of irreducible characters are

```
[ [ 1, 2 ], [ 11, 2 ], [ 12, 2 ], [ 54, 2 ], [ 55, 2 ], [ 66, 4 ],
  [ 120, 2 ], [ 132, 2 ], [ 154, 2 ], [ 165, 2 ], [ 220, 4 ], [ 275, 2 ],
  [ 297, 2 ], [ 320, 2 ], [ 330, 2 ], [ 440, 2 ], [ 462, 8 ], [ 495, 4 ],
  [ 528, 2 ], [ 540, 2 ], [ 594, 4 ], [ 616, 2 ], [ 792, 4 ], [ 891, 2 ],
  [ 924, 1 ], [ 945, 2 ], [ 990, 2 ], [ 1155, 2 ], [ 1320, 4 ], [ 1408, 2 ],
  [ 1440, 2 ], [ 1485, 6 ], [ 1584, 2 ], [ 1650, 2 ], [ 1728, 2 ],
  [ 1760, 4 ], [ 1925, 4 ], [ 1980, 2 ], [ 2079, 2 ], [ 2100, 1 ],
  [ 2112, 2 ], [ 2310, 4 ], [ 2376, 6 ], [ 2520, 2 ], [ 2640, 1 ],
  [ 2673, 2 ], [ 2772, 6 ], [ 2970, 2 ], [ 3024, 1 ], [ 3080, 2 ],
  [ 3168, 4 ], [ 3465, 4 ], [ 3520, 4 ], [ 3564, 2 ], [ 3696, 2 ],
  [ 3960, 6 ], [ 4158, 2 ], [ 4455, 2 ], [ 4620, 10 ], [ 4752, 6 ],
  [ 4950, 4 ], [ 5544, 6 ], [ 5632, 2 ], [ 5775, 2 ], [ 5940, 8 ],
  [ 6160, 4 ], [ 6600, 2 ], [ 6930, 6 ], [ 7128, 2 ], [ 7700, 1 ],
  [ 7920, 2 ], [ 8316, 10 ], [ 9240, 10 ], [ 9900, 6 ], [ 10395, 8 ],
  [ 10560, 8 ], [ 11088, 10 ], [ 11550, 8 ], [ 11880, 10 ], [ 12320, 6 ],
  [ 13200, 2 ], [ 13860, 12 ], [ 14256, 1 ], [ 14784, 4 ], [ 14850, 4 ],
  [ 15400, 2 ], [ 15840, 4 ], [ 16632, 8 ], [ 17325, 4 ], [ 18480, 9 ],
  [ 19008, 8 ], [ 19800, 6 ], [ 20790, 12 ], [ 21120, 2 ], [ 23100, 14 ],
  [ 23760, 4 ], [ 24640, 2 ], [ 26400, 4 ], [ 27720, 12 ], [ 28512, 2 ],
  [ 29568, 2 ], [ 29700, 8 ], [ 30800, 1 ], [ 31185, 4 ], [ 31680, 4 ],
  [ 34650, 10 ], [ 35640, 4 ], [ 36960, 6 ], [ 37422, 8 ], [ 41580, 17 ],
  [ 44352, 8 ], [ 44550, 2 ], [ 46200, 14 ], [ 47520, 8 ], [ 50688, 2 ],
  [ 51975, 4 ], [ 52800, 2 ], [ 55440, 10 ], [ 59400, 4 ], [ 62370, 2 ],
  [ 63360, 4 ], [ 66528, 8 ], [ 69300, 2 ], [ 71280, 4 ], [ 73920, 6 ],
  [ 74844, 1 ], [ 79200, 2 ], [ 83160, 14 ], [ 89100, 1 ], [ 92400, 1 ],
  [ 95040, 7 ], [ 99792, 2 ], [ 103950, 4 ], [ 110880, 4 ], [ 118272, 2 ],
  [ 133056, 2 ], [ 133650, 2 ], [ 138600, 4 ], [ 147840, 2 ], [ 166320, 2 ] ]
```

- Sizes of conjugacy classes are

[1, 66, 495, 924, 495, 66, 1, 132, 1320, 5940, 15840, 27720, 33264, 27720,
15840, 5940, 1320, 132, 5940, 5940, 95040, 166320, 166320, 665280, 415800,
415800, 665280, 166320, 166320, 95040, 5940, 5940, 110880, 332640,
1995840, 665280, 1663200, 4989600, 6652800, 2217600, 1663200, 4989600,
1995840, 665280, 110880, 332640, 831600, 4989600, 831600, 13305600,
13305600, 4989600, 29937600, 4989600, 13305600, 13305600, 831600, 4989600,
831600, 1995840, 19958400, 9979200, 19958400, 39916800, 3991680, 1995840,
19958400, 9979200, 332640, 332640, 9979200, 9979200, 665280, 1760, 15840,
63360, 147840, 221760, 221760, 147840, 63360, 15840, 1760, 126720, 126720,
887040, 887040, 2661120, 2661120, 4435200, 4435200, 4435200, 4435200,
2661120, 2661120, 887040, 887040, 126720, 126720, 2661120, 5322240,
2661120, 13305600, 26611200, 13305600, 26611200, 53222400, 26611200,
26611200, 53222400, 26611200, 13305600, 26611200, 13305600, 2661120,
5322240, 2661120, 17740800, 53222400, 53222400, 17740800, 53222400,
159667200, 159667200, 53222400, 53222400, 159667200, 159667200, 53222400,
17740800, 53222400, 53222400, 17740800, 26611200, 106444800, 159667200,
106444800, 26611200, 26611200, 106444800, 159667200, 106444800, 26611200,
591360, 591360, 7096320, 8870400, 8870400, 23654400, 8870400, 8870400,
7096320, 591360, 591360, 17740800, 17740800, 35481600, 141926400,
70963200, 70963200, 106444800, 106444800, 212889600, 141926400, 70963200,
70963200, 17740800, 17740800, 35481600, 106444800, 106444800, 425779200,
106444800, 106444800, 425779200, 425779200, 425779200, 425779200,
106444800, 106444800, 425779200, 106444800, 106444800, 70963200, 70963200,
425779200, 212889600, 212889600, 141926400, 31539200, 94617600, 283852800,
94617600, 94617600, 283852800, 94617600, 31539200, 189235200, 567705600,
567705600, 189235200, 567705600, 189235200, 189235200, 567705600,
63078400, 378470400, 63078400, 23760, 190080, 665280, 1330560, 1663200,
1330560, 665280, 190080, 23760, 1330560, 1330560, 7983360, 7983360,
19958400, 19958400, 26611200, 26611200, 19958400, 19958400, 7983360,
7983360, 1330560, 1330560, 19958400, 39916800, 19958400, 79833600,
159667200, 79833600, 119750400, 239500800, 119750400, 79833600, 159667200,
79833600, 19958400, 39916800, 19958400, 79833600, 239500800, 239500800,
79833600, 159667200, 479001600, 479001600, 159667200, 79833600, 239500800,
239500800, 79833600, 19958400, 19958400, 159667200, 239500800, 159667200,
39916800, 10644480, 10644480, 53222400, 53222400, 106444800, 106444800,
106444800, 106444800, 53222400, 53222400, 10644480, 10644480, 212889600,
212889600, 212889600, 212889600, 638668800, 638668800, 638668800,
638668800, 638668800, 638668800, 638668800, 638668800, 212889600,
212889600, 212889600, 212889600, 638668800, 638668800, 1277337600,
1277337600, 638668800, 638668800, 638668800, 638668800, 1277337600,
1277337600, 638668800, 638668800, 425779200, 851558400, 425779200,
851558400, 1703116800, 851558400, 425779200, 851558400, 425779200,
851558400, 1703116800, 851558400, 851558400, 1703116800, 851558400,
39916800, 39916800, 319334400, 239500800, 239500800, 319334400, 39916800,
39916800, 479001600, 479001600, 958003200, 1916006400, 958003200,
958003200, 479001600, 479001600, 958003200, 239500800, 239500800,
479001600, 1916006400, 479001600, 479001600, 1277337600, 1277337600,
2554675200, 2554675200, 1277337600, 1277337600, 319334400, 319334400,
1916006400, 304128, 2128896, 6386688, 10644480, 10644480, 6386688,
2128896, 304128, 12773376, 12773376, 63866880, 63866880, 127733760,
127733760, 127733760, 127733760, 63866880, 63866880, 12773376, 12773376,
127733760, 255467520, 127733760, 383201280, 766402560, 383201280,
383201280, 766402560, 383201280, 127733760, 255467520, 127733760,
255467520, 766402560, 766402560, 255467520, 255467520, 766402560,
766402560, 255467520, 85155840, 85155840, 340623360, 340623360, 510935040,
510935040, 340623360, 340623360, 85155840, 85155840, 1021870080,
1021870080, 1021870080, 1021870080, 2043740160, 2043740160, 2043740160,
2043740160, 1021870080, 1021870080, 1021870080, 1021870080, 1021870080,
1021870080, 2043740160, 2043740160, 1021870080, 1021870080, 1362493440,
2724986880, 1362493440, 1362493440, 2724986880, 1362493440, 510935040,
510935040, 1532805120, 1532805120, 1532805120, 1532805120, 510935040,
510935040, 3065610240, 3065610240, 3065610240, 3065610240, 3065610240,
3065610240, 3065610240, 3065610240, 4087480320, 4087480320, 4087480320,
4087480320, 1226244096, 1226244096, 4904976384, 1226244096, 1226244096,
2452488192, 2452488192, 4904976384, 3548160, 21288960, 53222400, 70963200,
53222400, 21288960, 3548160, 106444800, 106444800, 425779200, 425779200,
638668800, 638668800, 425779200, 425779200, 106444800, 106444800,
638668800, 1277337600, 638668800, 1277337600, 2554675200, 1277337600,
638668800, 1277337600, 638668800, 212889600, 212889600, 1277337600,
1277337600, 425779200, 567705600, 567705600, 1703116800, 1703116800,

1703116800, 1703116800, 567705600, 567705600, 3406233600, 3406233600,
3406233600, 3406233600, 3406233600, 3406233600, 3406233600, 3406233600,
2270822400, 4541644800, 2270822400, 2554675200, 2554675200, 5109350400,
5109350400, 2554675200, 2554675200, 2554675200, 2554675200, 5109350400,
5109350400, 5109350400, 8174960640, 8174960640, 8174960640, 8174960640,
3406233600, 3406233600, 6812467200, 36495360, 182476800, 364953600,
364953600, 182476800, 36495360, 729907200, 729907200, 2189721600,
2189721600, 2189721600, 2189721600, 729907200, 729907200, 2189721600,
4379443200, 2189721600, 2189721600, 4379443200, 2189721600, 2919628800,
2919628800, 5839257600, 5839257600, 2919628800, 2919628800, 5839257600,
5839257600, 5839257600, 5839257600, 8758886400, 8758886400, 8758886400,
8758886400, 14014218240, 14014218240, 319334400, 1277337600, 1916006400,
1277337600, 319334400, 3832012800, 3832012800, 7664025600, 7664025600,
3832012800, 3832012800, 1916006400, 1916006400, 7664025600, 3832012800,
10218700800, 10218700800, 10218700800, 10218700800, 7664025600,
7664025600, 15328051200, 2270822400, 6812467200, 6812467200, 2270822400,
13624934400, 13624934400, 13624934400, 13624934400, 18166579200,
18166579200, 12262440960, 24524881920, 12262440960, 12262440960,
12262440960, 24524881920, 44590694400, 44590694400, 40874803200,
40874803200]

- Sizes of centralizers of each conjugacy class (in same order) are

[980995276800, 14863564800, 1981808640, 1061683200, 1981808640,
14863564800, 980995276800, 7431782400, 743178240, 165150720, 61931520,
35389440, 29491200, 35389440, 61931520, 165150720, 743178240, 7431782400,
165150720, 165150720, 10321920, 5898240, 5898240, 1474560, 2359296,
2359296, 1474560, 5898240, 5898240, 10321920, 165150720, 165150720,
8847360, 2949120, 491520, 1474560, 589824, 196608, 147456, 442368, 589824,
196608, 491520, 1474560, 8847360, 2949120, 1179648, 196608, 1179648,
73728, 73728, 196608, 32768, 196608, 73728, 73728, 1179648, 196608,
1179648, 491520, 49152, 98304, 49152, 24576, 245760, 491520, 49152, 98304,
2949120, 2949120, 98304, 98304, 1474560, 557383680, 61931520, 15482880,
6635520, 4423680, 4423680, 6635520, 15482880, 61931520, 557383680,
7741440, 7741440, 1105920, 1105920, 368640, 368640, 221184, 221184,
221184, 221184, 368640, 368640, 1105920, 1105920, 7741440, 7741440,
368640, 184320, 368640, 73728, 36864, 73728, 36864, 18432, 36864, 36864,
18432, 36864, 73728, 36864, 73728, 368640, 184320, 368640, 55296, 18432,
18432, 55296, 18432, 6144, 6144, 18432, 18432, 6144, 6144, 18432, 55296,
18432, 18432, 55296, 36864, 9216, 6144, 9216, 36864, 36864, 9216, 6144,
9216, 36864, 1658880, 1658880, 138240, 110592, 110592, 41472, 110592,
110592, 138240, 1658880, 1658880, 55296, 55296, 27648, 6912, 13824, 13824,
9216, 9216, 4608, 6912, 13824, 13824, 55296, 55296, 27648, 9216, 9216,
2304, 9216, 9216, 2304, 2304, 2304, 2304, 9216, 9216, 2304, 9216, 9216,
13824, 13824, 2304, 4608, 4608, 6912, 31104, 10368, 3456, 10368, 10368,
3456, 10368, 31104, 5184, 1728, 1728, 5184, 1728, 5184, 5184, 1728, 15552,
2592, 15552, 41287680, 5160960, 1474560, 737280, 589824, 737280, 1474560,
5160960, 41287680, 737280, 737280, 122880, 122880, 49152, 49152, 36864,
36864, 49152, 49152, 122880, 122880, 737280, 737280, 49152, 24576, 49152,
12288, 6144, 12288, 8192, 4096, 8192, 12288, 6144, 12288, 49152, 24576,
49152, 12288, 4096, 4096, 12288, 6144, 2048, 2048, 6144, 12288, 4096,
4096, 12288, 49152, 49152, 6144, 4096, 6144, 24576, 92160, 92160, 18432,
18432, 9216, 9216, 9216, 9216, 18432, 18432, 92160, 92160, 4608, 4608,
4608, 4608, 1536, 1536, 1536, 1536, 1536, 1536, 1536, 1536, 4608, 4608,
4608, 4608, 1536, 1536, 1536, 768, 768, 1536, 1536, 1536, 1536, 768, 768, 1536,
1536, 2304, 1152, 2304, 1152, 576, 1152, 2304, 1152, 2304, 1152, 576,
1152, 1152, 576, 1152, 24576, 24576, 3072, 4096, 4096, 3072, 24576, 24576,
2048, 2048, 1024, 512, 1024, 1024, 2048, 2048, 1024, 4096, 4096, 2048,
512, 2048, 2048, 768, 768, 384, 384, 768, 768, 3072, 3072, 512, 3225600,
460800, 153600, 92160, 92160, 153600, 460800, 3225600, 76800, 76800,
15360, 15360, 7680, 7680, 7680, 7680, 15360, 15360, 76800, 76800, 7680,
3840, 7680, 2560, 1280, 2560, 2560, 1280, 2560, 7680, 3840, 7680, 3840,
1280, 1280, 3840, 3840, 1280, 1280, 3840, 11520, 11520, 2880, 2880, 1920,
1920, 2880, 2880, 11520, 11520, 960, 960, 960, 960, 480, 480, 480, 480,
960, 960, 960, 960, 960, 960, 480, 480, 960, 960, 720, 360, 720, 720, 360,
720, 1920, 1920, 640, 640, 640, 640, 1920, 1920, 320, 320, 320, 320, 320,
320, 320, 320, 240, 240, 240, 240, 800, 800, 200, 800, 800, 400, 400, 200,
276480, 46080, 18432, 13824, 18432, 46080, 276480, 9216, 9216, 2304, 2304,
1536, 1536, 2304, 2304, 9216, 9216, 1536, 768, 1536, 768, 384, 768, 1536,

```
768, 1536, 4608, 4608, 768, 768, 2304, 1728, 1728, 576, 576, 576, 576,
1728, 1728, 288, 288, 288, 288, 288, 288, 288, 288, 432, 216, 432, 384,
384, 192, 192, 384, 384, 384, 384, 192, 192, 192, 120, 120, 120, 120, 288,
288, 144, 26880, 5376, 2688, 2688, 5376, 26880, 1344, 1344, 448, 448, 448,
448, 1344, 1344, 448, 224, 448, 448, 224, 448, 336, 336, 168, 168, 336,
336, 168, 168, 168, 168, 112, 112, 112, 112, 70, 70, 3072, 768, 512, 768,
3072, 256, 256, 128, 128, 256, 256, 512, 512, 128, 256, 96, 96, 96, 96,
128, 128, 64, 432, 144, 144, 432, 72, 72, 72, 72, 54, 54, 80, 40, 80, 80,
80, 40, 22, 22, 24, 24 ]
```

- Orders of representatives of each conjugacy class (in same order) are

```
[ 1, 2, 2, 2, 2, 2, 2, 2, 4, 2, 4, 2, 4, 2, 4, 2, 4, 2, 2, 4, 4, 2, 4, 4, 2,
  4, 4, 2, 4, 4, 2, 4, 2, 4, 4, 4, 2, 4, 4, 4, 2, 4, 4, 4, 2, 4, 2, 4, 4, 4,
  4, 2, 4, 4, 4, 4, 2, 4, 4, 2, 4, 4, 4, 4, 4, 2, 4, 4, 2, 2, 4, 4, 4, 3, 6,
  6, 6, 6, 6, 6, 6, 6, 6, 6, 12, 6, 12, 6, 12, 6, 12, 6, 12, 6, 12, 6, 12,
  6, 12, 6, 12, 12, 6, 12, 12, 6, 12, 12, 6, 12, 12, 6, 12, 12, 6, 12, 12,
  6, 12, 12, 12, 6, 12, 12, 12, 6, 12, 12, 12, 6, 12, 12, 12, 6, 12, 12, 12,
  12, 6, 12, 12, 12, 12, 3, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 12, 6, 12,
  12, 6, 6, 12, 6, 12, 12, 6, 6, 12, 6, 6, 12, 12, 12, 6, 12, 12, 12, 6, 6,
  12, 12, 12, 6, 6, 12, 12, 12, 12, 3, 6, 6, 6, 6, 6, 6, 6, 6, 6, 12, 12, 6,
  6, 12, 12, 3, 6, 6, 4, 8, 4, 8, 4, 8, 4, 8, 4, 4, 8, 8, 4, 4, 8, 8, 4, 4,
  8, 8, 4, 4, 8, 4, 8, 4, 8, 4, 8, 4, 8, 4, 8, 4, 8, 4, 8, 4, 4, 8, 4, 8, 8,
  4, 8, 4, 4, 8, 4, 8, 4, 4, 8, 4, 8, 4, 12, 24, 24, 12, 12, 24, 24, 12, 12,
  24, 24, 12, 12, 24, 24, 12, 24, 12, 12, 24, 12, 24, 24, 12, 24, 12, 12,
  24, 12, 24, 24, 12, 12, 24, 24, 12, 12, 24, 24, 12, 12, 24, 12, 24, 12,
  24, 12, 24, 12, 12, 24, 12, 24, 12, 24, 4, 8, 8, 4, 8, 8, 4, 8, 4, 8, 8,
  8, 4, 8, 4, 8, 8, 4, 4, 8, 8, 4, 8, 12, 24, 24, 24, 12, 24, 4, 4, 8, 5,
  10, 10, 10, 10, 10, 10, 10, 10, 20, 10, 20, 10, 20, 10, 20, 10, 20, 10,
  20, 10, 20, 20, 10, 20, 20, 10, 20, 20, 10, 20, 20, 10, 20, 20, 20, 10,
  20, 20, 20, 15, 30, 30, 30, 30, 30, 30, 30, 30, 30, 30, 30, 60, 60, 30,
  30, 60, 60, 30, 30, 60, 60, 30, 30, 60, 60, 60, 60, 15, 30, 30, 30, 30,
  30, 20, 40, 20, 40, 20, 40, 20, 40, 20, 40, 20, 40, 20, 40, 20, 40, 60,
  120, 60, 120, 5, 10, 10, 10, 10, 10, 10, 20, 6, 12, 6, 12, 6, 12, 6, 6,
  12, 12, 12, 6, 12, 12, 12, 6, 12, 6, 12, 12, 12, 12, 12, 6, 12, 12, 6, 6,
  12, 12, 12, 6, 12, 12, 6, 6, 12, 12, 6, 6, 12, 12, 12, 12, 6, 12, 12, 6,
  12, 6, 12, 24, 12, 24, 12, 24, 12, 12, 24, 12, 24, 30, 60, 60, 30, 6, 6,
  12, 7, 14, 14, 14, 14, 14, 14, 28, 14, 28, 14, 28, 14, 28, 14, 28, 28, 14,
  28, 28, 21, 42, 42, 42, 42, 42, 42, 42, 84, 84, 28, 56, 28, 56, 35, 70, 8,
  16, 8, 16, 8, 8, 16, 16, 8, 8, 16, 8, 8, 16, 8, 24, 48, 48, 24, 8, 8, 16,
  9, 18, 18, 18, 18, 36, 18, 36, 9, 18, 10, 20, 10, 10, 10, 20, 11, 22, 12,
  12 ]
```

4.3 Random walk driven by the "constricted paired flips"

It is the random walk on $G_E \cong C_2^{11} \rtimes S_{12}$ parallel to the "constricted paired twists" on G_V. See former propositions.

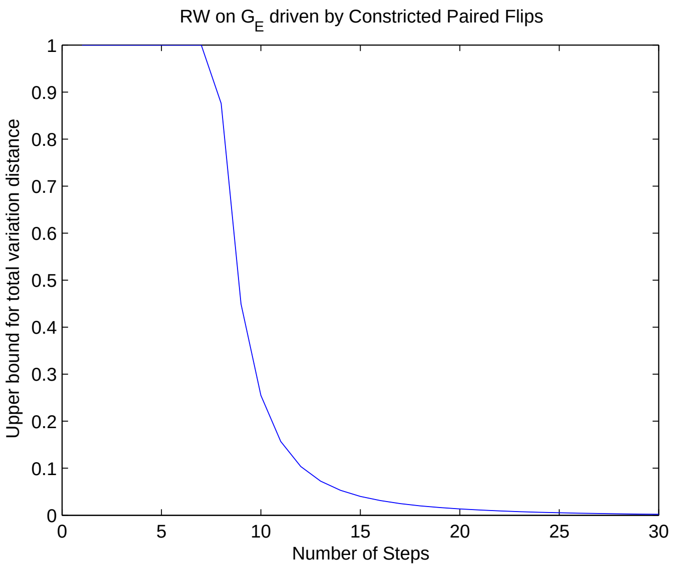
The sorted eigenvalues are

```
Columns 1 through 10
 -0.8333   -0.6806   -0.6667   -0.5556   -0.5278   -0.5278   -0.5278   -0.5000   -0.4583   -0.4167
Columns 11 through 20
 -0.4167   -0.4167   -0.4028   -0.3889   -0.3889   -0.3750   -0.3750   -0.3750   -0.3472   -0.3333
Columns 21 through 30
 -0.3333   -0.3333   -0.3333   -0.3333   -0.3333   -0.3056   -0.3056   -0.3056   -0.2917   -0.2778
Columns 31 through 40
 -0.2778   -0.2778   -0.2778   -0.2778   -0.2778   -0.2778   -0.2639   -0.2500   -0.2500   -0.2500
Columns 41 through 50
 -0.2500   -0.2500   -0.2500   -0.2361   -0.2361   -0.2361   -0.2222   -0.2222   -0.2222   -0.2222
Columns 51 through 60
 -0.2222   -0.2083   -0.2083   -0.2083   -0.1944   -0.1944   -0.1944   -0.1944   -0.1944   -0.1944
Columns 61 through 70
 -0.1806   -0.1806   -0.1806   -0.1806   -0.1806   -0.1667   -0.1667   -0.1667   -0.1667   -0.1667
Columns 71 through 80
 -0.1667   -0.1667   -0.1667   -0.1667   -0.1667   -0.1667   -0.1667   -0.1528   -0.1528   -0.1528
```

```
Columns 81 through 90
 -0.1528   -0.1389   -0.1389   -0.1389   -0.1389   -0.1389   -0.1389   -0.1389   -0.1389   -0.1389
Columns 91 through 100
 -0.1250   -0.1250   -0.1250   -0.1250   -0.1250   -0.1250   -0.1250   -0.1111   -0.1111   -0.1111
Columns 101 through 110
 -0.1111   -0.1111   -0.1111   -0.1111   -0.1111   -0.1111   -0.1111   -0.1111   -0.1111   -0.0972
Columns 111 through 120
 -0.0972   -0.0972   -0.0833   -0.0833   -0.0833   -0.0833   -0.0833   -0.0833   -0.0833   -0.0833
Columns 121 through 130
 -0.0833   -0.0833   -0.0833   -0.0833   -0.0833   -0.0833   -0.0833   -0.0833   -0.0833   -0.0833
Columns 131 through 140
 -0.0833   -0.0694   -0.0694   -0.0694   -0.0694   -0.0694   -0.0694   -0.0694   -0.0694   -0.0556
Columns 141 through 150
 -0.0556   -0.0556   -0.0556   -0.0556   -0.0556   -0.0556   -0.0556   -0.0556   -0.0556   -0.0556
Columns 151 through 160
 -0.0556   -0.0556   -0.0417   -0.0417   -0.0417   -0.0417   -0.0417   -0.0417   -0.0417   -0.0417
Columns 161 through 170
 -0.0417   -0.0417   -0.0417   -0.0417   -0.0417   -0.0417   -0.0417   -0.0278   -0.0278   -0.0278
Columns 171 through 180
 -0.0278   -0.0278   -0.0278   -0.0278   -0.0278   -0.0278   -0.0278   -0.0278   -0.0278   -0.0139
Columns 181 through 190
 -0.0139   -0.0139   -0.0139   -0.0139   -0.0139   -0.0139         0         0         0         0
Columns 191 through 200
       0         0         0         0         0         0         0         0         0         0
Columns 201 through 210
       0         0         0         0         0         0         0         0         0         0
Columns 211 through 220
       0         0         0         0         0    0.0139    0.0139    0.0139    0.0139    0.0139
Columns 221 through 230
  0.0139    0.0139    0.0139    0.0278    0.0278    0.0278    0.0278    0.0278    0.0278    0.0278
Columns 231 through 240
  0.0278    0.0278    0.0278    0.0278    0.0278    0.0278    0.0278    0.0278    0.0278    0.0278
Columns 241 through 250
  0.0278    0.0278    0.0278    0.0278    0.0278    0.0417    0.0417    0.0417    0.0417    0.0417
Columns 251 through 260
  0.0417    0.0417    0.0417    0.0417    0.0417    0.0417    0.0556    0.0556    0.0556    0.0556
Columns 261 through 270
  0.0556    0.0556    0.0556    0.0556    0.0556    0.0556    0.0556    0.0556    0.0556    0.0556
Columns 271 through 280
  0.0556    0.0556    0.0556    0.0556    0.0556    0.0556    0.0694    0.0694    0.0694    0.0694
Columns 281 through 290
  0.0694    0.0694    0.0694    0.0694    0.0694    0.0694    0.0694    0.0694    0.0833    0.0833
Columns 291 through 300
  0.0833    0.0833    0.0833    0.0833    0.0833    0.0833    0.0833    0.0833    0.0833    0.0833
Columns 301 through 310
  0.0833    0.0833    0.0833    0.0833    0.0833    0.0833    0.0833    0.0833    0.0833    0.0833
Columns 311 through 320
  0.0833    0.0972    0.0972    0.0972    0.0972    0.0972    0.0972    0.0972    0.0972    0.0972
Columns 321 through 330
  0.0972    0.0972    0.0972    0.1111    0.1111    0.1111    0.1111    0.1111    0.1111    0.1111
Columns 331 through 340
  0.1111    0.1111    0.1111    0.1111    0.1111    0.1111    0.1111    0.1111    0.1111    0.1111
Columns 341 through 350
  0.1111    0.1111    0.1111    0.1250    0.1250    0.1250    0.1250    0.1250    0.1250    0.1250
Columns 351 through 360
  0.1250    0.1250    0.1250    0.1250    0.1389    0.1389    0.1389    0.1389    0.1389    0.1389
Columns 361 through 370
  0.1389    0.1389    0.1389    0.1389    0.1389    0.1389    0.1389    0.1389    0.1389    0.1389
Columns 371 through 380
  0.1389    0.1389    0.1389    0.1389    0.1389    0.1389    0.1528    0.1528    0.1528    0.1528
Columns 381 through 390
  0.1528    0.1528    0.1528    0.1528    0.1667    0.1667    0.1667    0.1667    0.1667    0.1667
Columns 391 through 400
  0.1667    0.1667    0.1667    0.1667    0.1667    0.1667    0.1667    0.1667    0.1667    0.1667
Columns 401 through 410
  0.1667    0.1667    0.1667    0.1667    0.1667    0.1667    0.1667    0.1667    0.1667    0.1667
Columns 411 through 420
  0.1667    0.1667    0.1667    0.1806    0.1806    0.1806    0.1806    0.1806    0.1806    0.1806
Columns 421 through 430
  0.1944    0.1944    0.1944    0.1944    0.1944    0.1944    0.1944    0.1944    0.1944    0.1944
Columns 431 through 440
  0.1944    0.1944    0.2083    0.2083    0.2083    0.2083    0.2083    0.2083    0.2083    0.2083
Columns 441 through 450
  0.2083    0.2083    0.2083    0.2083    0.2083    0.2083    0.2083    0.2222    0.2222    0.2222
Columns 451 through 460
  0.2222    0.2222    0.2222    0.2222    0.2222    0.2222    0.2222    0.2222    0.2222    0.2222
Columns 461 through 470
  0.2361    0.2361    0.2361    0.2361    0.2361    0.2361    0.2361    0.2361    0.2500    0.2500
Columns 471 through 480
  0.2500    0.2500    0.2500    0.2500    0.2500    0.2500    0.2500    0.2500    0.2500    0.2500
Columns 481 through 490
  0.2500    0.2500    0.2500    0.2500    0.2500    0.2500    0.2500    0.2639    0.2639    0.2639
Columns 491 through 500
  0.2778    0.2778    0.2778    0.2778    0.2778    0.2778    0.2778    0.2778    0.2778    0.2778
Columns 501 through 510
  0.2778    0.2778    0.2917    0.2917    0.2917    0.2917    0.2917    0.2917    0.2917    0.3056
Columns 511 through 520
  0.3056    0.3056    0.3056    0.3056    0.3056    0.3056    0.3056    0.3056    0.3194    0.3194
Columns 521 through 530
  0.3194    0.3194    0.3333    0.3333    0.3333    0.3333    0.3333    0.3333    0.3333    0.3333
Columns 531 through 540
  0.3333    0.3333    0.3333    0.3333    0.3472    0.3472    0.3472    0.3472    0.3472    0.3611
Columns 541 through 550
  0.3611    0.3611    0.3611    0.3611    0.3611    0.3750    0.3750    0.3750    0.3889    0.3889
```

```
Columns 551 through 560
  0.3889    0.3889    0.3889    0.4028    0.4028    0.4028    0.4167    0.4167    0.4167    0.4167
Columns 561 through 570
  0.4167    0.4167    0.4306    0.4444    0.4444    0.4444    0.4444    0.4444    0.4444    0.4444
Columns 571 through 580
  0.4583    0.4722    0.4722    0.4722    0.5000    0.5000    0.5000    0.5000    0.5000    0.5000
Columns 581 through 590
  0.5139    0.5417    0.5417    0.5417    0.5556    0.5556    0.5694    0.5833    0.5833    0.5833
Columns 591 through 599
  0.6250    0.6667    0.6944    0.6944    0.6944    0.7222    0.8333    0.8472    1.0000
```

The upper bound for total variation by Upper Bound Lemma 7. Around 15 steps are enough to bring the total variation low.



5 The $3 \times 3 \times 3$ Rubik's Cube Group

5.1 Conjugacy classes and irreducible characters of G_3

The Rubik's cube group G_3 is an index-2 normal subgroup of $H = (C_3^7 \rtimes S_8) \times (C_2^{11} \rtimes S_{12})$. From previous two sections, we know the conjugacy classes in H can be characterized by pair of type-matrices $(a_V, a_E^{\pm})$, where a_V is the 3-by-8 type-matrix for $G_V = C_3^7 \rtimes S_8$ and $a_E^{\pm}$ the 2-by-12 type-matrix for $G_E = C_2^{11} \rtimes S_{11}$. Here if conjugacy class a_E in $C_2 \wr S_{12}$ splits into two classes in subgroup $C_2^{11} \rtimes S_{11}$, we denote them by a_E^+ and a_E^-.

Lemma 30. *Let $H = S_{n_1} \times S_{n_2} \times \cdots \times S_{n_r}$ be a r-fold direct product of symmetric groups and $G \triangleleft H$ be the index-2 normal subgroup of H such that all elements $(\sigma_1, \sigma_2, \ldots, \sigma_r) \in G$ satisfy $\prod_{i=1}^r sgn(\sigma_i) = 1$. Then a conjugacy class in H splits into two conjugacy classes of equal size in G iff the cycle lengths are pairwise different and odd for each $\sigma_i, 1 \leq i \leq r$.*

Proof. Similar to the proof for Lemma 1.2.10 (p12) of James and Kerber (1981) [6]. For $g = (\sigma_1, \ldots, \sigma_r) \in G$, we know the size of the conjugacy class in H is $|g^H| = \frac{|H|}{|C_H(g)|}$. Now

$|G| = \frac{1}{2}|H|$. $C_G(g) = C_H(g) \cap G$ is a subgroup of index 1 or 2 in $C_H(g)$ (if the centralizer contains $h_1, h_2, \ldots, h_r$ with total parity -1, then $h_1^2, h_1h_2, \ldots, h_1h_r$ have total parity 1 and are also in centralizer), i.e., either $C_G(g) = C_H(g)$ or $C_G(g)$ contains half the elements in $C_H(g)$. Thus *the conjugacy class splits iff* $C_H(g) \subseteq G$. Note for each σ_i, the centralizer always contains the idendity, an even permutation. Then the only way to have $C_H(g)$ to lie entirely in G (total parity is 1) is to force the centralizers of σ_i to contain only even permutations for all i. Consider one component σ_i. Recall the powers of a single cycle are in the centralizer of a permuation. If σ_i contains an even-length (2-,4-,6-,...) cycle, then the centralizer of σ_i contains odd permuations, e.g., power 1 of the even-length cycle. If σ_i contains only odd-length (1-,3-,5-,...) cycles and two cycles are of same length, then again the centralizer of σ_i contains odd permutations, e.g., the permutation mapping between these two cycles are an odd number of transpositions and thus is odd. Consier the last case: σ_i has cycles of pairwise different and odd lengths. Now σ_i's centralizer only contains powers of odd-length cycles, all of which are even permutations. Therefore g^H splits in G iff $C_H(g) \subseteq G$ iff all σ_is has cycles of pairwise different and odd lengths. □

Remark: Lemma 1.2.10 (p12) of James and Kerber (1981) [6] is the case $r = 1$, i.e., how the conjugacy classes in S_n split in A_n.

The character table of $H = (C_3^7 \rtimes S_8) \times (C_2^{11} \rtimes S_{12})$ is simply the tensor product of the character tables for $C_3^7 \rtimes S_8$ and $C_2^{11} \rtimes S_{12}$, respectively.

Question 31. *How do the irreducible characters of $H = (C_3^7 \rtimes S_8) \times (C_2^{11} \rtimes S_{12})$ split in the subgroup G_3?*

Lemma 32. *Let $G = S_{n_1} \times S_{n_2}$ and $H = \{A_{n_1} \times A_{n_2}\} \cup \{A_{n_1}^c \times A_{n_2}^c\}$ is the subgroup of index 2 in G such that the elements $(\rho_1, \rho_2) \in H$ satisfies $sgn(\rho_1) = sgn(\rho_2)$. The irreducible characters of $S_{n_1} \times S_{n_2}$ can be characterized by pair of partitions (α_1, α_2), where $\alpha_i, i = 1, 2$, is a partition of n_i. Then $\chi^{(\alpha_1,\alpha_2)} = \chi^{\alpha_1}\chi^{\alpha_2}$ splits in H iff $\alpha_1 = \alpha_1'$ and $\alpha_2 = \alpha_2'$.*

Proof. Apply Clifford theorem to normal subgroup of index 2, we know for an irreducible character χ of G, $\chi \downarrow H$ splits into sum of two irreducible characters of H iff χ is zero outside H. We know irreducible character χ^{α_i} of S_{n_i} splits in A_{n_i} iff $\alpha_i = \alpha_i'$ iff χ^{α_i} is 0 outside A_{n_i}. Note χ^{α_i} cannot be identically 0 on A_{n_i}, e.g., $\chi^{\alpha_i}(\text{id}) \geq 1$. Therefore $\chi^{(\alpha_1,\alpha_2)}$ is 0 on $H^c = (A_{n_1}^c \times A_{n_2}) \cup (A_{n_1} \times A_{n_2}^c)$ iff χ^{α_i} is 0 on $A_{n_i}^c$ for $i = 1, 2$ iff $\alpha_i = \alpha_i'$ for $i = 1, 2$. □

Now we generalize above results (split of conjugacy classes and irreducible characters) to our case $G = (C_3^7 \rtimes S_8) \times (C_2^{11} \rtimes S_{12})$.

Proposition 33. *A conjugacy class in $G \wr S_n$ splits in $G \wr A_n$ iff the type matrix is of form*

$$\begin{pmatrix} \leq 1 & 0 & \leq 1 & 0 & \ldots \\ \vdots & & \vdots & & \\ \leq 1 & 0 & \leq 1 & 0 & \ldots \end{pmatrix}$$

Proof. Kerber (1975) [9] 1.19 p15. □

By Corollary 5, we know the conjugacy classes of $C_3^7 \rtimes S_8$ can be parametrized by type matrix just as in $C_3 \wr S_8$ except we require the total product to be 1 (0 in C_3).

Corollary 34. *A conjugacy class with sign 1 in $C_3^7 \rtimes S_8$ splits in $C_3^7 \rtimes A_8$ iff the type matrix is of form*

$$\begin{pmatrix} \leq 1 & 0 & \leq 1 & 0 & \dots \\ \leq 1 & 0 & \leq 1 & 0 & \dots \\ \leq 1 & 0 & \leq 1 & 0 & \dots \end{pmatrix}.$$

Proof. If the type matrix $\{a_{ij}\}$ is of the specified form, then we know all centralizers of $\{a_{ij}\}$ in $C_3 \wr S_8$ lie in $C_3 \wr A_8$. So $|C_{C_3^7 \rtimes A_8}(a) = |C_{C_3^7 \rtimes S_8}(a)|$ and thus the class splits into 2. If the type matrix is not of the specified form, then we know there are centralizers of $\{a_{ij}\}$ in $C_3 \wr A_8^c$. Let $\{g_1, \dots, g_m\}$ be enumeration of centralizers in $C_3^7 \rtimes A_8^c$. $\{g_1^2, g_1g_2, \dots, g_1g_m\}$ are distinct centralizers in $C_3^7 \rtimes A_8$. So $C_{C_3^7 \rtimes A_8}(a)$ is of index 1 or 2 in $C_{C_3^7 \rtimes S_8}(a)$. It is 2 because we know there are centralizers in $C_3^7 \rtimes A_8^c$ (if $(\vec{w}, \rho) \in C_3 \wr A_8^c$ is any centralizer of $\{a_{ij}\}$, $(\vec{w}, \rho)^3 \in C_3^7 \rtimes A_8^c$ is too). ☐

By Lemma 23, we know the conjugacy classes in $C_2^{11} \rtimes S_{12}$ are parametrized by type matrix as in $C_2 \wr S_{12}$ but if the type matrix is of form

$$\begin{pmatrix} 0 & * & 0 & * & \dots \\ 0 & 0 & 0 & 0 & \dots \end{pmatrix}$$

then it splits into two in $C_2^{11} \rtimes S_{12}$.

Corollary 35. *A conjugacy class with sign 1 in $C_2^{11} \rtimes S_{12}$ splits in $C_2^{11} \rtimes A_{12}$ iff the type matrix is of form*

$$\begin{pmatrix} \leq 1 & 0 & \leq 1 & 0 & \dots \\ \leq 1 & 0 & \leq 1 & 0 & \dots \end{pmatrix} \tag{9}$$

or

$$\begin{pmatrix} 0 & 0 & 0 & 0 & \dots \\ 0 & * & 0 & * & \dots \end{pmatrix}. \tag{10}$$

In particular, the classes

$$\begin{pmatrix} 0 & * & 0 & * & \dots \\ 0 & 0 & 0 & 0 & \dots \end{pmatrix}^{\pm 1}$$

do not split in $C_2^{11} \rtimes A_{12}$.

Proof. If $(\vec{w}, \rho) \in C_2 \rtimes A_{12}$ has type matrix of form (9), then all its centralizers are in $C_2 \wr A_{12}$, therefore it splits. Similarly, if it has type matrix of form (10), then all its centralizers are in $C_2^{11} \rtimes A_{12}$ and $(C_2 \wr A_{12}^c) \cap (C_2^{11} \rtimes S_{12})^c$ (c.f. Kerber (1975) 1.27 p19 [9]). Thefore it splits. Now if it has type matrix not of these two forms, we know the number of centralizers in

$C_2^{11} \rtimes A_{12} \cup (C_2^{11} \rtimes S_{12})$ is equal to number of centralizers in $C_2^{11} \rtimes A_{12}^c \cap (C_2 \wr A_{12} \setminus C_2^{11} \rtimes S_{12})$. Also observer if $\{g_1, \ldots, g_m\}$ is enumeration of centralizers in $(C_2 \wr A_{12} \setminus C_2^{11} \rtimes S_{12})$, then $\{g_1^2, g_1 g_2, \ldots, g_1 g_m\}$ are distinct centralizers in $C_2^{11} \rtimes A_{12}$. So the number of centralizers in $C_2^{11} \rtimes A_{12}$ is greater than or equal to that in $(C_2 \wr A_{12} \setminus C_2^{11} \rtimes S_{12})$. Therefore there must be centralizers in $C_2^{11} \rtimes A_{12}^c$. Hence the class does not split. □

We know the Rubik's cube group is $G_3 = [(C_3^7 \rtimes A_8) \times (C_2^{11} \rtimes A_{12})] \cup [(C_3^7 \rtimes A_8^c) \times (C_2^{11} \rtimes A_{12}^c)]$ of index 2 in $G_V \times G_E$. Now observe that a conjugacy class $(\vec{w}, \rho, \vec{v}, \sigma)^{G_V \times G_E}$ splits in G_3 iff all centralizers of $(\vec{w}, \rho, \vec{v}, \sigma)$ lie in G_3 iff both $C_{C_3^7 \rtimes S_8}(\vec{w}, \rho) \subset C_3^7 \rtimes A_8$ and $C_{C_2^{11} \rtimes S_{12}}(\vec{v}, \sigma) \subset C_2^{11} \rtimes A_{12}$. The latter is true because if say $(\vec{w}', \rho')$ is centralizer of $(\vec{w}, \rho)$ and is not in $C_3^7 \rtimes A_8$, then $(\vec{w}', \rho', \vec{e}, 1_{S_{12}})$ will be an centralizer of $(\vec{w}, \rho, \vec{v}, \sigma)$ and not in G_3. Therefore

Corollary 36. *If conjugacy classes of $G_V \times G_E$ are parametrized by pair of type matrices (with possible ± 1 sign in the second type matrix), then a class splits over Rubik's cube group G_3 iff it is of form*

$$\left(\begin{pmatrix} \leq 1 & 0 & \leq 1 & 0 & \ldots \\ \leq 1 & 0 & \leq 1 & 0 & \ldots \\ \leq 1 & 0 & \leq 1 & 0 & \ldots \end{pmatrix}, \begin{pmatrix} \leq 1 & 0 & \leq 1 & 0 & \ldots \\ \leq 1 & 0 & \leq 1 & 0 & \ldots \end{pmatrix}\right) \tag{11}$$

or

$$\left(\begin{pmatrix} \leq 1 & 0 & \leq 1 & 0 & \ldots \\ \leq 1 & 0 & \leq 1 & 0 & \ldots \\ \leq 1 & 0 & \leq 1 & 0 & \ldots \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & \ldots \\ 0 & * & 0 & * & \ldots \end{pmatrix}\right). \tag{12}$$

Example 37. By Examples 3, 5 and 7, the moves "paired twists of 2 corner subcubes + paired flips of 2 edge subcubes" constitute a conjugacy class in the Rubik's cube group G_3. Size of this conjugacy class is $84 \times 132 = 11,088$.

Proposition 38. *The irreducible character $\chi^{\alpha_1, \alpha_2}$ of $C_2 \wr S_{12}$ splits in $C_2 \wr A_{12}$ iff $\alpha_1 = \alpha_1'$ and $\alpha_2 = \alpha_2'$. When it does not split, $\chi^{\alpha_1, \alpha_2}$ and $\chi^{\alpha_1', \alpha_2'}$ form an associated pair.*

Proof. Kerber (1975) [9] 2.25 p35. □

Proposition 39. *The irreducible character $\chi^{\alpha_1, \alpha_2, \alpha_3}$ of $C_3 \wr S_8$ splits in $C_3 \wr A_8$ iff $\alpha_i = \alpha_i', i = 1, 2, 3$. When it does not split, $\chi^{\alpha_1, \alpha_2, \alpha_3}$ and $\chi^{\alpha_1', \alpha_2', \alpha_3'}$ form an associated pair.*

Proof. ??? □

Corollary 40. *The irreducible character $\chi^{\alpha_1, \alpha_2, \alpha_3}$ of $C_3^7 \rtimes S_8$ splits in $C_3^7 \rtimes A_8$ iff $\alpha_i = \alpha_i', i = 1, 2, 3$. When it does not split, $\chi^{\alpha_1, \alpha_2, \alpha_3}$ and $\chi^{\alpha_1', \alpha_2', \alpha_3'}$ form an associated pair.*

Proof. If $\alpha_i = \alpha_i, i = 1, 2, 3$, $\chi^{\alpha_1, \alpha_2, \alpha_3}$ as a character of $C_3 \wr S_8$ is identically 0 on $C_3 \wr A_8^c$. So $\chi^{\alpha_1, \alpha_2, \alpha_3} \downarrow C_3^7 \rtimes S_8$ splits over $C_3^7 \rtimes A_8$ too by Clifford theory of normal subgroup of index 2. If $\alpha_i \neq \alpha_i'$ for some i, $\chi^{\alpha_1, \alpha_2, \alpha_3}$ and $\chi^{\alpha_1', \alpha_2', \alpha_3'}$ are associated pair in $C_3 \wr S_8$ wrt $C_3 \wr A_8$ (coincide on $C_3 \wr A_8$ and differs by a sign on $C_3 \wr A_8^c$). Suppose $\chi^{\alpha_1, \alpha_2, \alpha_3}$ is identically 0 on $C_3^7 \rtimes S_8 \cap C_3 \wr A_8^c$, then $\chi^{\alpha_1,\alpha_2,\alpha_3} \downarrow C_3^7 \rtimes S_8 = \chi^{\alpha_1',\alpha_2',\alpha_3'} \downarrow C_3^7 \rtimes S_8$, which implies that $(\alpha_1, \alpha_2, \alpha_3)$ is equal to $(\alpha_3', \alpha_1', \alpha_2')$ or $(\alpha_2', \alpha_3', \alpha_1')$. This is impossible by checking using a computer. Therefore $\chi^{\alpha_1,\alpha_2,\alpha_3}$ is nonzero for some element in $C_3^7 \rtimes S_8 \cap C_3 \wr A_8^c$ and thus does not split. □

The case for $C_2^{11} \rtimes A_{12}$ is slightly more complicated.

Corollary 41. *The irreducible characters of* $C_2^{11} \rtimes S_{12}$ *are* $\{\chi^{(\alpha_1,\alpha_2)} : \alpha_1 < \alpha_2\} \cup \{\chi_\pm^{(\alpha,\alpha)}\}$.
(a) An irreducible character $\chi^{(\alpha_1,\alpha_2)}$ *of* $C_2^{11} \rtimes S_{12}$ *where* $\alpha_1 < \alpha_2$ *splits in* $C_2^{11} \rtimes A_{12}$ *iff* $\alpha_i = \alpha_i', i = 1, 2$ *or* $(\alpha_1, \alpha_2) = (\alpha_2', \alpha_1')$.
(b) An irreducible character $\chi_\pm^{(\alpha,\alpha)}$ *of* $C_2^{11} \rtimes S_{12}$ *splits in* $C_2^{11} \rtimes A_{12}$ *iff* $\alpha = \alpha'$.

Proof. (a) For $\chi^{(\alpha_1,\alpha_2)}, \alpha_1 < \alpha_2$. If $\alpha_i = \alpha_i', i = 1, 2$, then $\chi^{(\alpha_1,\alpha_2)}$ as character of $C_2 \wr S_{12}$ is identical 0 on $C_2 \wr A_{12}$. Thus $\chi^{(\alpha_1,\alpha_2)} \downarrow C_2^{12} \rtimes S_{12}$ splits. If $(\alpha_1, \alpha_2) = (\alpha_2', \alpha_1')$, then $\chi^{(\alpha_1,\alpha_2)} \downarrow C_2^{11} \rtimes S_{12} = \chi^{(\alpha_2',\alpha_1')} \downarrow C_2^{11} \rtimes S_{12} = \chi^{(\alpha_1',\alpha_2')} \downarrow C_2^{11} \rtimes S_{12}$. But $\chi^{(\alpha_1,\alpha_2)}$ is also associated with $\chi^{(\alpha_1',\alpha_2')}$ wrt $C_2 \wr A_{12}$, therefore we know $\chi^{(\alpha_1,\alpha_2)}$ is identically 0 on $C_2^{11} \rtimes A_{12}^c$ therefore splits in $C_2^{11} \rtimes A_{12}$. Now for the remaining case, i.e., $\alpha_i \neq \alpha_i'$ for some i and $(\alpha_1, \alpha_2) \neq (\alpha_2', \alpha_1')$, then we know $\chi^{(\alpha_1,\alpha_2)}$ cannot be identically 0 on $C_2^{11} \rtimes A_{12}^c$ (otherwise contradicting with $(\alpha_1, \alpha_2) \neq (\alpha_2', \alpha_1')$). Therefore $\chi^{(\alpha_1,\alpha_2)}$ does not split.
(b) If $\alpha = \alpha'$, then $\chi^{(\alpha,\alpha)}$ is identically 0 on $C_2 \wr A_{12}$. Also note χ^α as character of S_6 is also 0 on all odd permutations. Therefore characters $\chi_{\pm 1}^{(\alpha,\alpha)}$ of $C_2^{11} \rtimes S_{12}$ is identically 0 on $C_2^{11} \rtimes A_{12}^c$ (c.f. formula ???). Therefore $\chi_\pm^{(\alpha,\alpha)}$ splits in $C_2^{11} \rtimes A_{12}$. If $\alpha \neq \alpha'$, then $\chi^{(\alpha,\alpha)}$ and $\chi^{(\alpha',\alpha')}$ restricted on $C_2^{11} \rtimes S_{12}$ are different. So $\chi^{(\alpha,\alpha)}$ is nonzero on some element in $C_2^{11} \rtimes A_{12}^c$. If this element is not in any split class $(2\pi, [])$, we are done. If it is in some split class, we know formula ??? can always take nonzero value on certain split class. In any case, we know $\chi_\pm^{(\alpha,\alpha)}$ does not split in $C_2^{11} \rtimes A_{12}$. □

Now we combine these two corollaries to get characterization of all the irreducible characters of the Rubik's cube group.

Proposition 42. *The characters* $\chi^{(\alpha_1,\alpha_2,\alpha_3)} \times \chi^{(\beta_1,\beta_2)}$ *of* $G_V \times G_E$ *split iff* $\alpha_i = \alpha_i'$ *for* $i = 1, 2, 3$ *and* $\beta_i = \beta_i', i = 1, 2$ *or* $(\beta_1, \beta_2) = (\beta_2', \beta_1')$. *When it does not split,* $\chi^{(\alpha_1',\alpha_2',\alpha_3')} \times \chi^{(\beta_1',\beta_2')}$ *form an associated pair.*

Proof. Similar to ???. □

Next question is how to compute the character table of Rubik's cube from that of $G_V \times G_E$. Specifically, how to compute character values of split characters on split classes? Use notation $\alpha = (\alpha_1, \alpha_2, \alpha_3), \alpha' = (\alpha_1', \alpha_2', \alpha_3'), \beta = (\beta_1, \beta_2), \beta' = (\beta_1', \beta_2')$.

We first study character values of $\chi^{(\alpha=\alpha',\beta=\beta')\pm}$. Let M denote the group $(C_3^7 \rtimes A_8) \times (C_2^{11} \rtimes A_{11})$ which is of index 2 in G_3, i.e., $M \lhd_2 G_3 \lhd_2 H_3$. Note $\chi^{(\alpha=\alpha',\beta=\beta')} \downarrow M = \chi^{\alpha+}\chi^{\beta+} + \chi^{\alpha+}\chi^{\beta-} + \chi^{\alpha-}\chi^{\beta+} + \chi^{\alpha-}\chi^{\beta-}$ where $\chi^{\alpha\pm}(\chi^{\beta\pm})$ are the split characters of $C_3^7 \rtimes S_8(C_2^{11} \rtimes S_{12})$ on $C_3^7 \rtimes A_8(C_2^{11} \rtimes A_{12})$ respectively. Everyone of these four irreducible constituents induces

$\chi^{(\alpha=\alpha',\beta=\beta')}$ in H_3, Frobenius' reciprocity law implies that two of them induce the irreducible constituents of $\chi^{(\alpha=\alpha',\beta=\beta')}$ in G_3. (???) Now consideration of representing matrices shows $\chi^{\alpha+}\chi^{\beta+}\uparrow G_3 \sim \chi^{\alpha-}\chi^{\beta-}\uparrow G_3$ and $\chi^{\alpha+}\chi^{\beta-}\uparrow G_3 \sim \chi^{\alpha-}\chi^{\beta+}\uparrow G_3$. Therefore

$$[\alpha]\sharp[\beta]\downarrow G_3 = [\alpha]^+\sharp[\beta]^+\uparrow G_3 + [\alpha]^+\sharp[\beta]^-\uparrow G_3 (= [\alpha]^-\sharp[\beta]^-\uparrow G_3 + [\alpha]^-\sharp[\beta]^+\uparrow G_3).$$

Now take a representative x of a split class of form (11). Then

$$\begin{aligned}[\chi^{(\alpha,\beta)+} - \chi^{(\alpha,\beta)-}](x) &= ([\alpha]^+\sharp[\beta]^+ - [\alpha]^+\sharp[\beta]^-)^{G_3}(x) \\ &= C_{G_3}(x)\sum_{y\sim x}\frac{1}{C_M(y)}([\alpha]^+\sharp[\beta]^+ - [\alpha]^+\sharp[\beta]^-)(y)\end{aligned}$$

Now if x is not in class $(\cdot, h(\beta))$, then $[\beta]^+$ and $[\beta]^-$ is same on y (???). Similarly if x is not in class $(h(\alpha),\cdot)$, then $[\alpha]^+$ and $[\alpha]^-$ is same on y (???). Therefore we know $\chi^{(\alpha,\beta)+} = \chi^{(\alpha,\beta)-}$ on classes besides $(h(\alpha), h(\beta))^{\pm}$. Note elements in this class must have total sign 1. Now let x be any representative of class $(h(\alpha), h(\beta))^+$. We know this class splits in M into two classes denoted by $(h(\alpha)^+, h(\beta)^+)$ and $(h(\alpha)^-, h(\beta)^-)$. Thus the sum is over two representatives y_1, y_2 and $C_M(y_1) = C_M(y_2) = C_{G_3}(x)$ and hence

$$\begin{aligned}&[\chi^{(\alpha,\beta)+} - \chi^{(\alpha,\beta)-}](x) = (\chi^{\alpha+}\chi^{\beta+} - \chi^{\alpha+}\chi^{\beta-})[(h(\alpha)^+, h(\beta)^+) + (h(\alpha)^-, h(\beta)^-)] \\ = \ &(\chi^{\alpha+}_{h(\alpha)^+} - \chi^{\alpha-}_{h(\alpha)^+})(\chi^{\beta+}_{h(\beta)^+} - \chi^{\beta-}_{h(\beta)^+}).\end{aligned}$$

Similarly, for $x \in (h(\alpha), h(\beta))^+$, we have

$$[\chi^{(\alpha,\beta)+} - \chi^{(\alpha,\beta)-}](x) = -(\chi^{\alpha+}_{h(\alpha)^+} - \chi^{\alpha-}_{h(\alpha)^+})(\chi^{\beta+}_{h(\beta)^+} - \chi^{\beta-}_{h(\beta)^+}).$$

Therefore, we have

Proposition 43. *For $(\alpha,\beta) = (\alpha',\beta')$, the character values of the split characters $\chi^{(\alpha,\beta)\pm}$ on split classes $(h(\alpha), h(\beta))\pm$ are*

$$\chi^{(\alpha,\beta)\delta}_{(h(\alpha),h(\beta))\epsilon} = \frac{1}{2}[\chi^{(\alpha,\beta)}_{(h(\alpha),h(\beta))} + \epsilon\delta(\chi^{\alpha+}_{h(\alpha)^+} - \chi^{\alpha-}_{h(\alpha)^+})(\chi^{\beta+}_{h(\beta)^+} - \chi^{\beta-}_{h(\beta)^+})] \tag{13}$$

whereas the value of $\chi^{(\alpha,\beta)\pm}$ on other classes are 1/2 of original values.

Next we study split characters $\chi^{(\alpha,\beta)\pm}$ where $\alpha = \alpha', (\beta_1, \beta_2) = (\beta_2', \beta_1')$. ???

Now it only remains to derive character tables of $C_3^7 \rtimes A_8$ and $C_2^{11} \rtimes A_{12}$, i.e., know the correction terms in formula (13).

Let $\chi^{(\beta_1=\beta_1',\beta_2=\beta_2')}$ be an irreducible character of $C_2 \wr S_{12}$ that splits on $C_2 \wr A_{12}$. Kerber (1975) [9] p42 describes the two irreducible constituents as

$$\begin{aligned}&\left(1^{\tilde{s}}\epsilon^t\right) \otimes ([\alpha]^+\sharp[\beta]^+\uparrow S_s\times S_t\cap A_n)')^{S_2\wr A_n}; \\ &\left(1^{\tilde{s}}\epsilon^t\right) \otimes ([\alpha]^+\sharp[\beta]^-\uparrow S_s\times S_t\cap A_n)')^{S_2\wr A_n}.\end{aligned}$$

Let x be an representative of a split class (two rows podd)+/-. Note this conjugacy class does not split in the subgroup $C_2 \wr (S_s \times S_t \cap A_n)$.

$$\begin{aligned}&[\chi^{(\alpha_1,\alpha_2)+} - \chi^{(\alpha_1,\alpha_2)-}](x) = \left(1^{\tilde{s}}\epsilon^t\right) \otimes ([\alpha]^+\sharp[\beta]^+\uparrow S_s\times S_t\cap A_n)' - [\alpha]^+\sharp[\beta]^-\uparrow S_s\times S_t\cap A_n)')^{S_2\wr A_n}(x) \\ =\end{aligned}$$

5.2 Random walk on Rubik's cube driven by "paired twists + paired flips"

Consider this random walk on Rubik's cube group G_3. At each step, with 1/2 probability do nothing, with 1/2 probability do paired twists + paired flips simultaneously. Note we cannot do only one of them because we need them to have same sign. This probability measure is

$$P(\vec{v},\sigma,\vec{w},\rho) = \begin{cases} \frac{1}{2} & \text{if } \vec{v}=\vec{e}_8, \sigma = 1_{S_8}, \vec{w}=\vec{e}_{12}, \rho = 1_{S_{12}} \\ \frac{1}{2}\frac{1}{3\binom{8}{2}2\binom{12}{2}} & \text{if } (\vec{v},\sigma) \text{ is a paired twists, } (\vec{w},\rho) \text{ is a paired flips.} \end{cases} \quad (14)$$

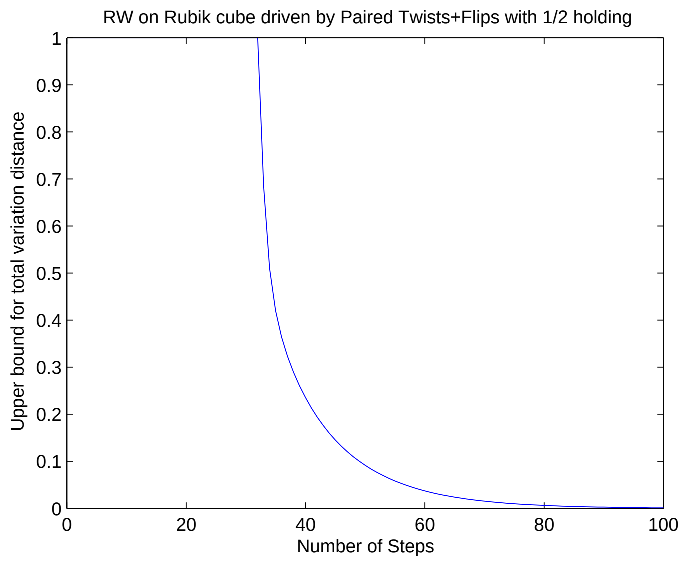
with support on conjugacy classes

$$\left(\begin{pmatrix} 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right) \quad (15)$$

and

$$\left(\begin{pmatrix} 6 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 10 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right) \quad (16)$$

of sizes 1 and 11088 respectively. We have the largest eigenvalue $11/12 \approx 0.9167$. 50 steps are enough to bring the total variation down. (Not surprising since we have 1/2 holding, the number doubled.)



5.3 Random walk on $(C_3^7 \rtimes A_8) \times (C_2^{11} \rtimes A_{12})$ driven by "3 cycles"

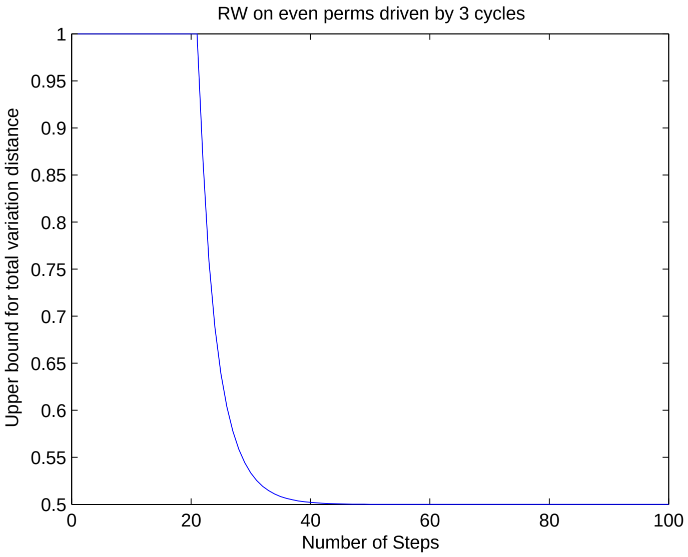
Consider this walk: at each step we do a 3-cycle on corner subcubes with probability 1/2 otherwise do a 3-cycle on edge subcubes. This walk has support on conjugacy classes

$$\left(\begin{pmatrix} 5 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right) \tag{17}$$

and

$$\left(\begin{pmatrix} 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 9 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right) \tag{18}$$

of sizes 108864 and 1760 respectively. Note these classes only generate the subgroup $(C_3^7 \rtimes A_8) \times (C_2^{11} \rtimes A_{12})$. So the total variation approaches 1/2.

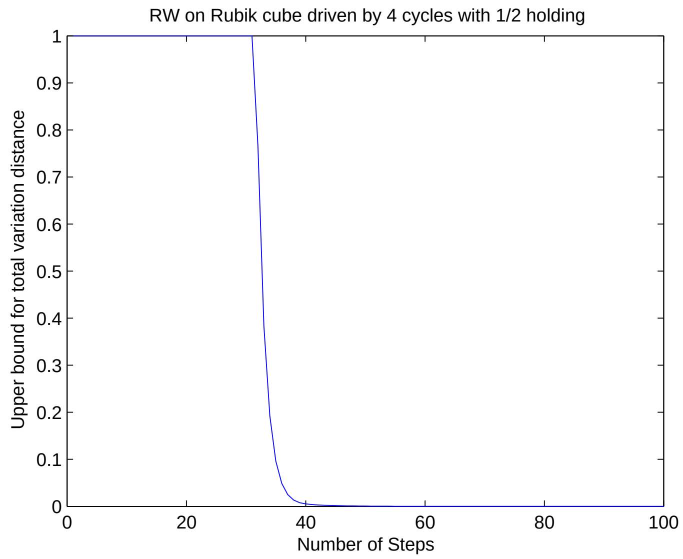


5.4 Random walk on Rubik's cube driven by "4 cycles" with 1/2 holding

The conjugacy class the generators live in is of form

$$\left(\begin{pmatrix} 4 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 8 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right) \tag{19}$$

and has size 269438400. Each step we do nothing with probability 1/2 or multiply an element uniformly chosen from this class.



5.5 Comparison theorem for random walk driven by 12 quarter turns

References

[1] Diaconis, P. (1988). *Group Representations in Probability and Statistics*, IMS, Hayward.

[2] Diaconis, P., and Shahshahani, M. (1981). Generating a random permutation with random transopositions, *Z. Wahrsch. Verw. Gebiete*, **57**, 159-179.

[3] Diaconis, P., Stroock, D. (1991). Geometric bounds for eigenvalues of markov chains, *Ann. Appl. Prob.*, **1**, 36-61.

[4] Dummit, D. S. and Foote, R. M. (2004). *Abstract Algebra*, 3rd Ed., John Wiley & Sons, Inc.

[5] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.4*; 2005, (`http://www.gap-system.org`).

[6] James, G. and Kerber A. (1981). *The Representation Theory of the Symmetric Group*, Addison-Wesley.

[7] James, G. and Liebeck M. (2001). *Representations and Characters of Groups*, 2nd Ed., Cambridge University Press.

[8] Joyner, W. D. *Mathematics of the Rubik's Cube*, unpublished, available at

`http://web.usna.navy.mil/~wdj/books.html`

[9] Kerber, A. (1975) *Representations of Permutation Groups II*, Springer-Verlag.

[10] Pfeiffer, G. (1994). Character Tables of Weyl Groups in GAP, *Bayreuther Mathematiche Schriften*, **47**, 165-222.

[11] Saloff-Coste, L. (1996). Lectures on finite Markov chains, *Lectures on Probability Theory and Statistics*, P. Bernard, editor, Springer.

[12] Saloff-Coste, L. (2004). Random walks on finite groups, *Probability on Discrete Structures*, H. Kesten, editor, Springer.

[13] Schoolfield Jr, C. H. (2002). Random walks on wreath products of groups, *J. Theor. Prob.*, **15**, 667-693.

[14] Serre, P. (1971). *Linear Representations of Finite Groups*, Springer.

[15] Weisstein, E. S. Rubik's cube, From *MathWorld*–A Wolfram Web Resource.

`http://mathworld.wolfram.com/RubiksCube.html`