ST552, Homework 4

Due Wednesday, Oct 9, 2013 (extended to Oct 16)

1. Fill out the missing steps in our derivation of the best (minimum variance) affine unbiased estimator (MVAUE) for the Aiken model: $\mathrm{E}(\boldsymbol{y}) = \boldsymbol{X}\boldsymbol{b}$, $\mathrm{Cov}(\boldsymbol{y}) = \sigma^2 \boldsymbol{V}$, assuming $\boldsymbol{V}$ is positive definite.

   (a) Show that if $\boldsymbol{V}$ is positive definite, then $\boldsymbol{V}^{-1}$ is positive definite.

   (b) If $\boldsymbol{V}$ is positive definite, show that $\mathcal{C}(\boldsymbol{X}^T\boldsymbol{V}^{-1}\boldsymbol{X}) = \mathcal{C}(\boldsymbol{X}^T)$.

   (c) Prove that the minimum trace affine unbiased estimator (MTAUE) is unique by showing that

   $$\boldsymbol{A} = \boldsymbol{\Lambda}(\boldsymbol{X}^T\boldsymbol{V}^{-1}\boldsymbol{X})^{-}\boldsymbol{X}^T\boldsymbol{V}^{-1}$$

   is invariant to the choice of the generalized inverse $(\boldsymbol{X}^T\boldsymbol{V}^{-1}\boldsymbol{X})^{-}$. (Hint: you may use the fact there exists a positive definite matrix $\boldsymbol{V}^{1/2}$ such that $\boldsymbol{V} = \boldsymbol{V}^{1/2}\boldsymbol{V}^{1/2}$.)

2. Fill out the missing steps in our derivation of the best (minimum variance) affine unbiased estimator (MVAUE) for the Aiken model with linear constraints $\boldsymbol{R}\boldsymbol{b} = \boldsymbol{r}$. We assume $\boldsymbol{V} \succ \boldsymbol{0}_{n\times n}$ and let $\boldsymbol{G} = \boldsymbol{X}^T\boldsymbol{V}^{-1}\boldsymbol{X} + \boldsymbol{R}^T\boldsymbol{R}$.

   (a) Show that $\boldsymbol{G}$ is positive semidefinite and $\mathcal{C}(\boldsymbol{G}) \supset \mathcal{C}(\boldsymbol{X}^T) \cup \mathcal{C}(\boldsymbol{R}^T)$.

   (b) Show that $\mathcal{C}(\boldsymbol{R}\boldsymbol{G}^{-}\boldsymbol{R}^T) \supset \mathcal{C}(\boldsymbol{R}\boldsymbol{G}^{-}\boldsymbol{X}^T)$ and thus $\boldsymbol{R}\boldsymbol{G}^{-}\boldsymbol{R}^T(\boldsymbol{R}\boldsymbol{G}^{-}\boldsymbol{R}^T)^{-}\boldsymbol{R}\boldsymbol{G}^{-}\boldsymbol{\Lambda}^T = \boldsymbol{R}\boldsymbol{G}^{-}\boldsymbol{\Lambda}^T$.

   (c) Check that the variance of the MTAUE is

   $$\mathrm{Cov}(\widehat{\boldsymbol{\Lambda}\boldsymbol{b}}) = \sigma^2\boldsymbol{\Lambda}\boldsymbol{G}^{-}\boldsymbol{\Lambda}^T - \sigma^2\boldsymbol{\Lambda}\boldsymbol{G}^{-}\boldsymbol{R}^T(\boldsymbol{R}\boldsymbol{G}^{-}\boldsymbol{R}^T)^{-}\boldsymbol{R}\boldsymbol{G}^{-}\boldsymbol{\Lambda}^T.$$

   (d) Show that the MTAUE is also the MVAUE.

3. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two matrices of same number of rows. Show that the following statements are equivalent. (Hint: I followed the route $(b) \to (a) \to (h), (i) \to (d), (f) \to (c), (e) \to (g) \to (b)$.)

   (a) $\mathcal{C}(\boldsymbol{A}) \cap \mathcal{C}(\boldsymbol{B}) = \{\boldsymbol{0}\}$

   (b) $\mathrm{rank}(\boldsymbol{A}\boldsymbol{A}^T + \boldsymbol{B}\boldsymbol{B}^T) = \mathrm{rank}(\boldsymbol{A}) + \mathrm{rank}(\boldsymbol{B})$

   (c) $\boldsymbol{A}^T(\boldsymbol{A}\boldsymbol{A}^T + \boldsymbol{B}\boldsymbol{B}^T)^{-}\boldsymbol{A}$ is idempotent

   (d) $\boldsymbol{A}^T(\boldsymbol{A}\boldsymbol{A}^T + \boldsymbol{B}\boldsymbol{B}^T)^{-}\boldsymbol{A} = \boldsymbol{A}^{-}\boldsymbol{A}$

   (e) $\boldsymbol{B}^T(\boldsymbol{A}\boldsymbol{A}^T + \boldsymbol{B}\boldsymbol{B}^T)^{-}\boldsymbol{B}$ is idempotent

   (f) $\boldsymbol{B}^T(\boldsymbol{A}\boldsymbol{A}^T + \boldsymbol{B}\boldsymbol{B}^T)^{-}\boldsymbol{B} = \boldsymbol{B}^{-}\boldsymbol{B}$

   (g) $\boldsymbol{A}^T(\boldsymbol{A}\boldsymbol{A}^T + \boldsymbol{B}\boldsymbol{B}^T)^{-}\boldsymbol{B} = \boldsymbol{0}$

   (h) $(\boldsymbol{A}\boldsymbol{A}^T + \boldsymbol{B}\boldsymbol{B}^T)^{-}$ is a generalized inverse of $\boldsymbol{A}\boldsymbol{A}^T$

   (i) $(\boldsymbol{A}\boldsymbol{A}^T + \boldsymbol{B}\boldsymbol{B}^T)^{-}$ is a generalized inverse of $\boldsymbol{B}\boldsymbol{B}^T$

4. In class we showed that the MVAUE for Aitken model (non-singular $\boldsymbol{V}$) with linear constraints $\boldsymbol{Rb} = \boldsymbol{r}$ is

$$\widehat{\boldsymbol{\Lambda b}} = \boldsymbol{\Lambda G}^- \boldsymbol{X}^T \boldsymbol{V}^{-1} \boldsymbol{y} + \boldsymbol{\Lambda G}^- \boldsymbol{R}^T (\boldsymbol{RG}^- \boldsymbol{R}^T)^- (\boldsymbol{r} - \boldsymbol{RG}^- \boldsymbol{X}^T \boldsymbol{V}^{-1} \boldsymbol{y})$$

where $\boldsymbol{G} = \boldsymbol{X}^T \boldsymbol{V}^{-1} \boldsymbol{X} + \boldsymbol{R}^T \boldsymbol{R}$, and has variance

$$\text{Cov}(\widehat{\boldsymbol{\Lambda b}}) = \sigma^2 \boldsymbol{\Lambda G}^- \boldsymbol{\Lambda}^T - \sigma^2 \boldsymbol{\Lambda G}^- \boldsymbol{R}^T (\boldsymbol{RG}^- \boldsymbol{R}^T)^- \boldsymbol{RG}^- \boldsymbol{\Lambda}^T.$$

Show that in the special case $\mathcal{C}(\boldsymbol{X}^T) \cap \mathcal{C}(\boldsymbol{R}^T) = \{\boldsymbol{0}\}$, the MVAUE simplifies to

$$\widehat{\boldsymbol{\Lambda b}} = \boldsymbol{\Lambda G}^- (\boldsymbol{X}^T \boldsymbol{V}^{-1} \boldsymbol{y} + \boldsymbol{R}^T \boldsymbol{r}),$$

with variance matrix

$$\text{Cov}(\widehat{\boldsymbol{\Lambda b}}) = \sigma^2 \boldsymbol{\Lambda G}^- \boldsymbol{\Lambda}^T - \sigma^2 \boldsymbol{\Lambda G}^- \boldsymbol{R}^T \boldsymbol{RG}^- \boldsymbol{\Lambda}^T.$$

(Hint: Use Q3 with $\boldsymbol{A} = \boldsymbol{X}^T \boldsymbol{V}^{-1/2}$ and $\boldsymbol{B} = \boldsymbol{R}^T$.)

5. (Generalized inverse of bordered Gramian matrix) Let $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ be a positive semidefinite matrix and $\boldsymbol{B} \in \mathbb{R}^{n \times m}$. Let $\boldsymbol{N} = \boldsymbol{A} + \boldsymbol{BB}^T$ and $\boldsymbol{C} = \boldsymbol{B}^T \boldsymbol{N}^- \boldsymbol{B}$. Show the following facts.

   (a) $\mathcal{C}(\boldsymbol{A}) \subset \mathcal{C}(\boldsymbol{N})$, $\mathcal{C}(\boldsymbol{B}) \subset \mathcal{C}(\boldsymbol{N})$, $\mathcal{C}(\boldsymbol{N}) = \mathcal{C}((\boldsymbol{A}, \boldsymbol{B}))$

   (b) $\boldsymbol{NN}^- \boldsymbol{A} = \boldsymbol{A}$, $\boldsymbol{NN}^- \boldsymbol{B} = \boldsymbol{B}$

   (c) $\mathcal{C}(\boldsymbol{B}^T) = \mathcal{C}(\boldsymbol{C})$, $\text{rank}(\boldsymbol{B}) = \text{rank}(\boldsymbol{C})$

   (d) A generalized inverse of the *bordered Gramian matrix*

   $$\boldsymbol{Z} = \begin{pmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{B}^T & \boldsymbol{0} \end{pmatrix}$$

   is

   $$\boldsymbol{Z}^- = \begin{pmatrix} \boldsymbol{N}^- - \boldsymbol{N}^- \boldsymbol{BC}^- \boldsymbol{B}^T \boldsymbol{N}^- & \boldsymbol{N}^- \boldsymbol{BC}^- \\ \boldsymbol{C}^- \boldsymbol{B}^T \boldsymbol{N}^- & -\boldsymbol{C}^- + \boldsymbol{CC}^- \end{pmatrix},$$

   where $\boldsymbol{N} = \boldsymbol{A} + \boldsymbol{BB}^T$ and $\boldsymbol{C} = \boldsymbol{B}^T \boldsymbol{N}^- \boldsymbol{B}$, and

   $$\boldsymbol{ZZ}^- = \begin{pmatrix} \boldsymbol{NN}^- & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{CC}^- \end{pmatrix}.$$

   (e) In the special case, $\mathcal{C}(\boldsymbol{B}) \subset \mathcal{C}(\boldsymbol{A})$, the generalized inverse in (d) takes a simpler form

   $$\boldsymbol{Z}^- = \begin{pmatrix} \boldsymbol{A}^- - \boldsymbol{A}^- \boldsymbol{B\Lambda}^- \boldsymbol{B}^T \boldsymbol{A}^- & \boldsymbol{A}^- \boldsymbol{B\Lambda}^- \\ \boldsymbol{\Lambda}^- \boldsymbol{B}^T \boldsymbol{A}^- & -\boldsymbol{\Lambda}^- \end{pmatrix},$$

   where $\boldsymbol{\Lambda} = \boldsymbol{B}^T \boldsymbol{A}^- \boldsymbol{B}$, and

   $$\boldsymbol{ZZ}^- = \begin{pmatrix} \boldsymbol{AA}^- & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{B}^- \boldsymbol{B} \end{pmatrix}.$$

   (Hint: it's easier to verify directly instead of deriving from (d).)