# ST552, Homework 8

## Due Mon, Dec 2, 2013

1. Derive the score vector (gradient), observed information matrix, and expected (Fisher) information matrix for the variance components model

$$\boldsymbol{y} \sim N(\boldsymbol{X}\boldsymbol{b}, \boldsymbol{V}),$$

where

$$\boldsymbol{V} = \sum_{i=0}^{m} \sigma_i^2 \boldsymbol{V}_i$$

with all $\boldsymbol{V}_i$ positive semidefinite and $\boldsymbol{V}$ nonsingular.

2. Derive the MLE of the variance components for the balanced one-way ANOVA random effects model (JM 8.4 p203).

3. Derive the REML of the variance components for the balanced one-way ANOVA random effects model (JM 8.5 p203).

4. Derive the method of moment estimates for the *unbalanced* one-way ANOVA random effects model, and then specialize to the balanced case.

5. (This is a 2013 Basic Exam Question) Consider the following linear model for response $Y_{ij}$:

$$Y_{ij} = \beta_0 + \boldsymbol{I}(j = 2)\beta_1 + b_i + e_{ij}, \quad i = 1, \ldots, n, j = 1, 2,$$

where $\beta_0$ and $\beta_1$ are constants, $b_i \sim N(0, \sigma_b^2)$, and $e_{ij} \sim N(0, \sigma_e^2)$ are independent to each other.

   (a) Find the ordinary least squares (OLS) estimates of $\beta_0$ and $\beta_1$. (Hint: Write the above model in matrix notation $\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$, where $\boldsymbol{Y} = (Y_{11}, Y_{12}, Y_{21}, Y_{22}, \ldots, Y_{n1}, Y_{n2})^T$. You need to be careful in defining $\boldsymbol{\epsilon}$.)

   (b) Assuming $\sigma_b^2$ and $\sigma_e^2$ are known, will the OLS estimates of $\beta_0$ and $\beta_1$ have smallest variances among all linearly unbiased estimates of $\beta_0$ and $\beta_1$?

   (c) Find the distributions of the OLS estimates $\beta_0$ and $\beta_1$ assuming $\sigma_b^2$ and $\sigma_e^2$ are known.

   (d) Define $Z_i = Y_{i2} - Y_{i1}$. Find an unbiased estimate of $\sigma_e^2$ based on $Z_i$s. Then construct a moment estimate of $\sigma_b^2$. (Hint: Express $E(Y_{ij}^2)$ in terms of $\sigma_b^2$ and $\sigma_e^2$, with $\beta_0$ and $\beta_1$ replaced by their OLS estimates.)

6. (This is another 2013 Basic Exam Question) Consider the following linear model for response $Y_{ij}$:

$$Y_{ij} = \alpha_i + \boldsymbol{I}(j = 2)\gamma + e_{ij}, \quad i = 1, \ldots, n, j = 1, 2,$$

where $\alpha_i$s ($i = 1, \ldots, n$) and $\gamma$ are constants, and $e_{ij} \sim N(0, \sigma_e^2)$ are independent residual errors.

(a) Find the ordinary least squares (OLS) estimates of $\alpha_i$s ($i = 1, \dots, n$) and $\gamma$. (Hint: Write the above model in matrix notation $\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$, where you may define $\boldsymbol{Y} = (Y_{11}, Y_{12}, Y_{21}, Y_{22}, \dots, Y_{n1}, Y_{n2})^T$.)

(b) Find the MLE of $\sigma_e^2$. It is acceptable to express the MLE in terms of the OLS estimates of $\alpha_i$s, $\gamma$, and the data.

(c) Find an unbiased estimator of $\sigma_e^2$ based on the sums of squared OLS residuals.

(d) Find the limit of the ratio of the estimators in (b) and (c) as $n \to \infty$.