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Monte Carlo Comparison of ANOVA, MIVQUE, REML, and ML Estimators of Variance Components

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For the one-way classification random model with unbalanced data, we compare five estimators of σ_a^2 and σ_e^2, the among- and within-treatments variance components: analysis of variance (ANOVA), maximum likelihood (ML), restricted maximum likelihood (REML), and two minimum variance quadratic unbiased (MIVQUE) estimators. MIVQUE(0) is MIVQUE with a priori values $\tilde{\sigma}_a^2 = 0$ and $\tilde{\sigma}_e^2 = 1$; MIVQUE(A) is MIVQUE with the ANOVA estimates used as a priori's. We enforce nonnegativity for all estimators, setting any negative estimate to zero in accord with usual practice. The estimators are compared through their biases and MSE's, estimated by Monte Carlo simulation. Our results indicate that the ANOVA estimators perform well, except with seriously unbalanced data when $\sigma_a^2/\sigma_e^2 > 1$; ML is excellent when $\sigma_a^2/\sigma_e^2 < 0.5$, and MIVQUE(A) is adequate; further iteration to the REML estimates is unnecessary. When $\sigma_a^2/\sigma_e^2 \geq 1$, MIVQUE(0) (the default for SAS's PROCEDURE VARCOMP) is poor for estimating σ_a^2 and very poor for σ_e^2, even for just mildly unbalanced data.

KEY WORDS: Iterative MIVQUE; Maximum likelihood; Minimum variance estimation; Restricted maximum likelihood.

1. INTRODUCTION

Variance components are commonly estimated in the course of determining appropriate sampling designs, establishing quality control procedures, or, in statistical genetics, estimating heritabilities and genetic correlations. Traditionally, the estimators used most often have been the analysis of variance (ANOVA) estimators, which are obtained by equating observed and expected mean squares from an analysis of variance and solving the resulting equations. (See, e.g., Searle 1971, ch. 9.) If the data are balanced (that is, if they have equal numbers of observations in corresponding subclasses), the ANOVA estimators have many appealing properties: they are unbiased, minimum variance among all unbiased estimators that are quadratic functions of the observations, and, under normality, minimum variance among all unbiased estimators. However, with unbalanced data (those with unequal subclass frequencies), all of these properties are lost except unbiasedness.

Because variance components must often be estimated from unbalanced data, research has been directed toward estimation methods whose properties do not depend on balanced data, and that provide general unifying criteria for variance components estimation. Two general classes of estimators have sparked considerable interest: maximum likelihood and restricted maximum likelihood (ML and REML), and minimum norm and minimum variance quadratic unbiased estimation (MINQUE and MIVQUE). In fact, there are strong links between these two classes of estimators, as is illustrated in Section 2.

Harville (1977) reviews the literature on the properties, advantages, and disadvantages of ML and REML. Briefly, while ML estimates the variance components by those values which maximize the full likelihood function over the parameter space, REML partitions the likelihood into two pieces, one of which is free of the fixed effects, and maximizes only *that* portion of the likelihood. In contrast to ML, REML takes into account the loss in degrees of freedom associated with estimation of fixed effects. In practice, a principal limitation of ML and REML is that the variance components estimates must usually be obtained iteratively, and the required computing may be difficult and perhaps infeasible for large data sets.

Swallow and Searle (1978) summarize Rao's (1971b) derivation of MIVQUE. To compute the MIVQUE's the user must supply a priori values for the variance components. The estimators are then functions of the data and of the a priori values, and are only locally minimum variance; that is, they are minimum variance only when each a priori value equals the true value of the corresponding variance component. Realistically, the user cannot provide perfect a priori values, so, in application, the estimators will not be minimum variance. Notwithstanding the lack of perfect values, Swallow (1981) has shown that for the one-way classification random model under normality (with σ_a^2 and σ_e^2 the between- and within-treatments variance components, respectively), if $\sigma_a^2/\sigma_e^2 > 1$ and the a priori estimate of σ_a^2/σ_e^2 is not a drastic underestimate, the MIVQUE of σ_a^2 is often more efficient than the ANOVA estimator; the MIVQUE and ANOVA estimators of σ_e^2 usually differ little.

Under normality (assumed in the Monte Carlo comparisons that follow) the MIVQUE and MINQUE equations are identical. Specifying the a priori values of the variance components for MIVQUE is equivalent to specifying the particular Euclidean norm (specified as "weights") to be minimized for MINQUE (Rao 1971a,1972). When the data are balanced, both the a priori values of MIVQUE and the weights of MINQUE drop out, and the MIVQUE, MINQUE, and ANOVA estimators are identical. With unbalanced data, MIVQUE and MINQUE suffer from computing difficulties similar to those of ML and REML, except in some special cases where the estimators are available in forms that do not require matrix inversion (see, for example, Brocklebank 1981, Swallow and Searle 1978).

The presence of "a priori" values of the variance components in the MIVQUE estimators with unbalanced data suggests the possibility of iteration, using the "estimates" from each round of iteration as the "a priori" values in the next round, and repeating until the "estimates" converge (Harville 1969, Rao 1972,1979). When such iteration is used, the estimators become biased, since the "a priori" values are functions of the data after the first iteration. We will call the iterated estimators I-MIVQUE's and I-MINQUE's in keeping with Brown's (1976) nomenclature, but meaning only that the MIVQUE and MINQUE equations are used iteratively—the MIVQUE and MINQUE properties are not preserved.

Under normality, the equations for I-MIVQUE, I-MINQUE, and REML are identical, with the initial a priori values of MIVQUE, the initial weights of MINQUE, and the starting point for iteration with REML playing analogous roles. A key difference, for REML, however, is that the equations are solved subject to a nonnegativity constraint, whereas the theory of I-MIVQUE and I-MINQUE allows negative estimates. Nonnegativity constraints are discussed further in Section 2.

With so many available estimators of variance components from unbalanced data, which one should the prospective user choose? Are the ANOVA estimators still a viable alternative, even though their optimal properties are lost under unbalanced data? How do they compare with estimators that have stronger theoretical support but are much more demanding computationally? To shed light on these questions, we present here a side-by-side Monte Carlo comparison of the ANOVA, ML, REML, and two MIVQUE estimators under the one-way classification random model with unbalanced data. This direct comparison provides the user with some basis for selecting an estimator.

2. THE MODEL AND ESTIMATORS TO BE COMPARED

Under the one-way classification random model assuming normality, y_{ij}, the jth observation in the ith group, can be expressed as

$$y_{ij} = \mu + a_i + e_{ij}, \tag{1}$$

where μ is an unknown parameter, a_i and e_{ij} are mutually independent random variables from normal distributions with zero means and variances σ_a^2 and σ_e^2, respectively, $i = 1, \ldots, a$ with $a \geq 2$, $j = 1, \ldots, n_i$ with $n_i \geq 1$ for all i and $n_i > 1$ for some i, and $N = \sum n_i$. The variance components to be estimated are σ_a^2 and σ_e^2. The n-pattern or vector of subgroup sizes is $(n_1, n_2, \ldots, n_a)$, and the data are called unbalanced whenever the n_i are not all equal. When the y_{ij} are ordered by i and by j within i, (1) may be rewritten in matrix notation as

$$\mathbf{y} = \mathbf{1}_N\mu + \begin{bmatrix} \mathbf{1}_{n_1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_{n_2} & & \vdots \\ \vdots & & \ddots & \vdots \\ \mathbf{0} & \cdots & \cdots & \mathbf{1}_{n_a} \end{bmatrix} \mathbf{a} + \mathbf{I}_N\mathbf{e}$$

$$= \mathbf{X}\mu + \mathbf{Z}_1\mathbf{a} + \mathbf{Z}_2\mathbf{e},$$

where $\mathbf{a}' = (a_1, a_2, \ldots, a_a)$, $\mathbf{e}' = (e_{11}, e_{12}, \ldots, e_{an_a})$, $\mathbf{X} = \mathbf{1}_N$ (an N-vector with all elements unity), $\mathbf{Z}_1$ is the $N \times a$ block-diagonal matrix shown above, and $\mathbf{Z}_2 = \mathbf{I}_N$. Thus $\mathbf{y}$ is a vector of random variables with mean $\mathbf{1}_N\mu$ and variance-covariance matrix

$$\mathbf{V} = \sigma_a^2\mathbf{V}_1 + \sigma_e^2\mathbf{V}_2, \tag{2}$$

where $\mathbf{V}_1 = \mathbf{Z}_1\mathbf{Z}_1'$ and $\mathbf{V}_2 = \mathbf{Z}_2\mathbf{Z}_2'$. When σ_a^2 and σ_e^2 are replaced in (2) by "estimates" $\tilde{\sigma}_a^2$ and $\tilde{\sigma}_e^2$ (perhaps a priori values), we have

$$\tilde{\mathbf{V}} = \tilde{\sigma}_a^2\mathbf{V}_1 + \tilde{\sigma}_e^2\mathbf{V}_2. \tag{3}$$

Finally, we define

$$\tilde{\mathbf{P}} = \tilde{\mathbf{V}}^{-1}[\mathbf{I} - \mathbf{X}(\mathbf{X}'\tilde{\mathbf{V}}^{-1}\mathbf{X})^{-1}\mathbf{X}'\tilde{\mathbf{V}}^{-1}]. \tag{4}$$

Five methods of estimating the variance components are considered.

2.1 Analysis of Variance (ANOVA)

The ANOVA estimators are obtained by equating the among- and within-groups mean squares from an analysis of variance to their expectations, and solving the resulting equations for $\hat{\sigma}_e^2$ and $\hat{\sigma}_a^2$. This gives

$$\hat{\sigma}_e^2 = \frac{\sum_i \sum_j (y_{ij} - \bar{y}_{i\cdot})^2}{(N - a)}$$

and

$$\hat{\sigma}_a^2 = \frac{\left[\sum_i n_i(\bar{y}_{i\cdot} - \bar{y}_{\cdot\cdot})^2 - (a - 1)\hat{\sigma}_e^2\right]}{\left(N - \sum_i n_i^2/N\right)}$$

where $\bar{y}_{i\cdot}$ and $\bar{y}_{\cdot\cdot}$ are the ith group mean and grand mean, respectively (Searle 1971, p. 474).

2.2 Maximum Likelihood (ML)

The ML estimates can be obtained by iterative solution of the two equations in $\hat{\sigma}_a^2$ and $\hat{\sigma}_e^2$ given by

$$\begin{bmatrix} \operatorname{tr}(\tilde{\mathbf{V}}^{-1}\mathbf{Z}_1\mathbf{Z}_1'\tilde{\mathbf{V}}^{-1}\mathbf{Z}_1\mathbf{Z}_1') & \operatorname{tr}(\tilde{\mathbf{V}}^{-1}\mathbf{Z}_1\mathbf{Z}_1'\tilde{\mathbf{V}}^{-1}\mathbf{Z}_2\mathbf{Z}_2') \\ \operatorname{tr}(\tilde{\mathbf{V}}^{-1}\mathbf{Z}_1\mathbf{Z}_1'\tilde{\mathbf{V}}^{-1}\mathbf{Z}_2\mathbf{Z}_2') & \operatorname{tr}(\tilde{\mathbf{V}}^{-1}\mathbf{Z}_2\mathbf{Z}_2'\tilde{\mathbf{V}}^{-1}\mathbf{Z}_2\mathbf{Z}_2') \end{bmatrix} \cdot \begin{bmatrix} \hat{\sigma}_a^2 \\ \hat{\sigma}_e^2 \end{bmatrix} = \begin{bmatrix} \mathbf{y}'\tilde{\mathbf{P}}\mathbf{Z}_1\mathbf{Z}_1'\tilde{\mathbf{P}}\mathbf{y} \\ \mathbf{y}'\tilde{\mathbf{P}}\mathbf{Z}_2\mathbf{Z}_2'\tilde{\mathbf{P}}\mathbf{y} \end{bmatrix}. \tag{5}$$

We use the ANOVA estimates as the starting point for iteration, that is, as the initial values for $\tilde{\sigma}_a^2$ and $\tilde{\sigma}_e^2$ in $\tilde{\mathbf{V}}$ and $\tilde{\mathbf{P}}$. Thereafter, $\tilde{\sigma}_a^2$ and $\tilde{\sigma}_e^2$ for each iteration are $\hat{\sigma}_a^2$ and $\hat{\sigma}_e^2$, respectively, from the previous iteration, and we continue iterating until the convergence criterion is satisfied. Following Herbach (1959), when $\hat{\sigma}_a^2 = 0$ we recalculate $\hat{\sigma}_e^2$ as though the N observations are a sample from a single population; this has little effect on $\hat{\sigma}_e^2$ unless $\sigma_a^2 \approx 0$. Additional details are provided below and in Section 3.

2.3 Restricted Maximum Likelihood (REML)

The two equations in $\hat{\sigma}_a^2$ and $\hat{\sigma}_e^2$ solved iteratively for the REML estimates are given by

$$\begin{bmatrix} \operatorname{tr}(\tilde{\mathbf{P}}\mathbf{Z}_1\mathbf{Z}_1'\tilde{\mathbf{P}}\mathbf{Z}_1\mathbf{Z}_1') & \operatorname{tr}(\tilde{\mathbf{P}}\mathbf{Z}_1\mathbf{Z}_1'\tilde{\mathbf{P}}\mathbf{Z}_2\mathbf{Z}_2') \\ \operatorname{tr}(\tilde{\mathbf{P}}\mathbf{Z}_1\mathbf{Z}_1'\tilde{\mathbf{P}}\mathbf{Z}_2\mathbf{Z}_2') & \operatorname{tr}(\tilde{\mathbf{P}}\mathbf{Z}_2\mathbf{Z}_2'\tilde{\mathbf{P}}\mathbf{Z}_2\mathbf{Z}_2') \end{bmatrix} \begin{bmatrix} \hat{\sigma}_a^2 \\ \hat{\sigma}_e^2 \end{bmatrix} = \begin{bmatrix} \mathbf{y}'\tilde{\mathbf{P}}\mathbf{Z}_1\mathbf{Z}_1'\tilde{\mathbf{P}}\mathbf{y} \\ \mathbf{y}'\tilde{\mathbf{P}}\mathbf{Z}_2\mathbf{Z}_2'\tilde{\mathbf{P}}\mathbf{y} \end{bmatrix}. \tag{6}$$

Again we use the ANOVA estimates as the starting point for iteration. It is noteworthy that equations (5) and (6) have identical right sides; their left sides differ only in that $\tilde{\mathbf{V}}^{-1}$ of (5) is replaced by $\tilde{\mathbf{P}}$ in (6). For the detailed algebra leading to (5) and (6), see Searle (1979).

2.4 MIVQUE With the ANOVA Estimates as A Priori Values (MIVQUE(A))

The MIVQUE(A) estimators use equation (6) without iteration, inserting the ANOVA estimates for $\tilde{\sigma}_a^2$ and $\tilde{\sigma}_e^2$ in $\tilde{\mathbf{P}}$ of (6). Although MIVQUE theory specifies that the a priori values be independent of the data (for unbiasedness of MIVQUE), users with no a priori values in mind often have suggested using some easily obtained estimates, such as the ANOVA estimates, as a priori values. The MIVQUE(A)'s are the first iterates of the REML estimators of Section 2.3.

2.5 MIVQUE With A Priori Values $\tilde{\sigma}_a^2 = 0$, $\tilde{\sigma}_e^2 = 1$ (MIVQUE(0))

MIVQUE(0) estimators are defined as those obtained from (6) without iteration, using $\tilde{\sigma}_a^2 = 0$, $\tilde{\sigma}_e^2 = 1$. These estimates are important by virtue of their inclusion as the default estimators in SAS's PROCEDURE VARCOMP (SAS *User's Guide: Statistics*, pp. 223–228); SAS calls them "MIVQUE0" estimators. In general, SAS's MIVQUE0 sets $\tilde{\sigma}_e^2 = 1$ and all other a priori values to zero, which simplifies the MIVQUE expressions and thus the required computing. However, the SAS MIVQUE0 user should be mindful that, in choosing to use MIVQUE0, a priori values are specified passively, but just as surely as with any other specification.

We have chosen to enforce nonnegativity in estimation of σ_a^2 for all estimators, which means that any negative MIVQUE(A), MIVQUE(0), or ANOVA estimate of σ_a^2 is set to zero. (The estimates of σ_e^2 are always nonnegative.) Enforcing nonnegativity violates the theory of MIVQUE and ANOVA estimation, since negative estimates must be allowed in order to retain unbiasedness, but it is in tune with common practice, in that users virtually always set negative estimates to zero, at least when estimating variance components individually (versus in linear combinations). In enforcing nonnegativity we forfeit unbiasedness in the MIVQUE(0) and ANOVA estimators, the only estimators of our five that would otherwise have been unbiased. The ANOVA estimate of σ_a^2 used as $\tilde{\sigma}_a^2$ in the ML, REML, and MIVQUE(A) calculations of Sections 2.2–2.4 is thus guaranteed to be nonnegative.

ML and REML estimates of variance components are by definition nonnegative. However, negative values can crop up in the iterative process and, if used as $\tilde{\sigma}_a^2$ in $\tilde{V}$, cause a singularity or near singularity that can disrupt computation. Therefore, after each iteration, we set any negative $\hat{\sigma}_a^2$ to zero before using it as $\tilde{\sigma}_a^2$ in the next iteration; doing so eliminated computational problems (such as inverting singular matrices) that we had occasionally observed in earlier work. A number of papers have dealt with the problems caused by negative values in iterating to ML or REML estimates (Corbeil and Searle 1976a,b, Harville 1977, Hemmerle and Hartley 1973, Hocking and Kutner 1975, Jennrich and Sampson 1976, Miller 1979). REML, as defined here, is identical to I-MIVQUE and I-MINQUE with nonnegativity enforced after each iteration.

Both ML and REML were allowed up to 20 iterations to converge. Convergence was said to have occurred when the estimates at the kth and $(k + 1)$th iterations satisfied

$$\frac{|\hat{\sigma}^2_{a,k+1} - \hat{\sigma}^2_{a,k}|}{1 + \hat{\sigma}^2_{a,k}} + \frac{|\hat{\sigma}^2_{e,k+1} - \hat{\sigma}^2_{e,k}|}{1 + \hat{\sigma}^2_{e,k}} < .0001.$$

The choice of convergence criterion is always arbitrary. Our selection was based on the premise that relative discrepancy is more meaningful than absolute discrepancy; unity was added to each denominator (a) to rule out division by zero, and (b) to prevent the convergence criterion from being overly stringent when either $\hat{\sigma}^2_{a,k}$ or $\hat{\sigma}^2_{e,k}$ was very small.

Defining ML, REML, MIVQUE(A), and MIVQUE(0) estimation using Equations (5) and (6) emphasizes their interrelationships. Using (5) and (6) iteratively as described to obtain the ML and REML estimates amounts to maximizing the likelihoods by Fisher's method of scoring (Harville 1977, Hocking and Kutner 1975, Rao 1979).

3. MONTE CARLO COMPARISON

Comparing variance components estimators for unbalanced data requires comparison under a variety of n-patterns and true values of the components (σ^2_a, σ^2_e). The n-patterns used in this study are listed in Table 1 and are among those used by Swallow and Searle (1978) and Swallow (1981). We have kept the n-pattern numbers (P_1, P_2, etc.) as assigned in the earlier papers for ease in cross-referencing; pattern numbers that are missing in Table 1 (P_3, P_6, P_{10}) are for patterns not discussed in detail in this article (because they gave results similar to those for P_2, P_5, and P_9, respectively). For a discussion of the interrelationships of these n-patterns, the rationale for their selection, and the difficulties in selecting n-patterns for comparing variance components estimators, see Swallow and Searle (1978, pp. 268–270). Patterns P_7 and P_{11} are included because previous experience has shown them to be especially troublesome ("worst cases") for variance components estimation. In considering values of (σ^2_a, σ^2_e) for comparing variance components estimators, one need only be concerned with the ratio σ^2_a/σ^2_e; we take $\sigma^2_a/\sigma^2_e = 0$, .1, .2, .5, 1.0, 2.0, and 5.0 (taking $\sigma^2_e \equiv 1$, so $\sigma^2_a/\sigma^2_e = \sigma^2_a$) for a fairly broad range of values.

For each n-pattern and value of σ^2_a/σ^2_e, 10,000 samples (or replicates) were generated. Given σ^2_a/σ^2_e and the n-pattern, the subgroup means and subgroup sums of squares are sufficient for the variance components estimators. (Individual observations need not be generated.) This was exploited in the Monte Carlo simulation, using Marsaglia and Bray's (1964) modified polar method for generating normal random variables (subgroup means) and Cheng and Feast's (1980) extension of Kinderman and Monahan's (1977) ratio-of-uniforms algorithm for generating chi-squared random variables (for subgroup sums of squares). The initial seed was a function of the n-pattern, set and updated using Schrage's (1979) portable Fortran version of the Lewis, Goodman, and Miller (1969) uniform pseudorandom number generator. All computations were in double-precision Fortran.

For each replication we estimated both σ^2_a and σ^2_e using all five estimators, and from these estimates we calculated mean squared error and bias. When either an ML or a REML estimator failed to converge in 20 iterations, as occasionally occurred, that replicate was dropped for all estimators, so that estimators would be compared for the same replicates.

4. RESULTS AND DISCUSSION

Underlying this study is a factorial structure consisting of four factors: the estimator used, the variance component being estimated, the value of σ^2_a/σ^2_e, and the n-pattern. In addition, the set of n-patterns has a fractional factorial structure of its own. The factors do interact, and the discussion focuses largely on these interactions.

4.1 Biases in Estimators of σ^2_a

Table 1 gives estimated biases of the estimators of σ^2_a for each estimator, n-pattern, and value of σ^2_a/σ^2_e. Each entry in Table 1 is computed from the approximately 10,000 usable replicates. For $\sigma^2_a > 0$, tabled values for which the estimated relative bias exceeds 10 percent (i.e., $|\text{bias}/\sigma^2_a| \geq .10$) are italicized for emphasis.

The most striking feature of Table 1 is the bias in the ML estimator. As noted earlier, ML fails to take account of loss in degrees of freedom associated with estimation of fixed effects (e.g., ML uses "a" rather than "$a - 1$" as the among-groups degrees of freedom). As a consequence, the ML estimator is generally biased downward, perhaps considerably so when a is small. In Table 1, when $\sigma^2_a/\sigma^2_e = \sigma^2_a$ is small, the bias is not serious. However, for larger values of σ^2_a, the ML estimator of σ^2_a has mean of approximately $[(a - 1)/a]\sigma^2_a$, which is to say that the downward bias is roughly $(1/a)\sigma^2_a$. To demonstrate this, we include in Table 1 ML-adj, which is $a/(a - 1)$ times the ML estimator; the bias remaining in ML-adj is very small. As a becomes large, the bias in the ML estimator becomes negligible. Approximate bias correction and ML-adj are introduced mainly for use in our dis-

Table 1. *Estimated Biases of Estimators of σ_a^2*

n-pattern		$\sigma_a^2/\sigma_e^2 = \sigma_a^2\ (\sigma_e^2 \equiv 1)$ .0	.1	.2	.5	1.0	2.0	5.0
$P_1 = (3, 5, 7)$	ANOVA	.082	***.063***	***.053***	.042	.018	−.009	.005
	MIVQUE(0)	.078	***.058***	***.049***	.041	.019	−.022	−.028
	MIVQUE(A)	.084	***.067***	***.056***	.041	.014	−.001	.026
	REML = I-MIVQUE	.083	***.066***	***.056***	.043	.018	.006	.036
	ML	.034	***−.020***	***−.068***	***−.190***	***−.383***	***−.731***	***−1.72***
	ML-adj	.051	***.019***	−.002	−.035	−.074	−.096	−.080
$P_2 = (1, 5, 9)$	ANOVA	.096	***.078***	***.063***	.048	.054	.033	.017
	MIVQUE(0)	.078	***.066***	***.056***	.043	.061	.046	.017
	MIVQUE(A)	.110	***.094***	***.075***	***.053***	.029	−.045	−.106
	REML = I-MIVQUE	.112	***.098***	***.083***	***.078***	.077	.036	.035
	ML	.036	***−.021***	***−.073***	***−.202***	***−.389***	***−.771***	***−1.79***
	ML-adj	.054	***.019***	−.009	***−.053***	−.083	−.156	−.185
$P_4 = (3, 3, 5, 5, 7, 7)$	ANOVA	.055	***.034***	.019	.008	−.005	.004	.029
	MIVQUE(0)	.052	***.032***	.018	.008	−.010	.003	.045
	MIVQUE(A)	.056	***.036***	***.020***	.006	−.007	−.002	.004
	REML = I-MIVQUE	.054	***.035***	***.020***	.008	−.002	.006	.014
	ML	.033	−.009	***−.043***	***−.110***	***−.205***	***−.366***	***−.859***
	ML-adj	.039	.009	−.012	−.032	−.046	−.039	−.031
$P_5 = (1, 1, 5, 5, 9, 9)$	ANOVA	.056	***.029***	***.023***	.017	.009	.013	.020
	MIVQUE(0)	.046	***.025***	***.021***	.020	.014	.013	.022
	MIVQUE(A)	.057	***.035***	***.030***	.015	−.014	−.041	−.091
	REML = I-MIVQUE	.052	***.031***	***.028***	.026	.015	.017	.015
	ML	.028	***−.018***	***−.046***	***−.112***	***−.213***	***−.388***	***−.895***
	ML-adj	.033	***−.001***	−.015	−.034	−.056	−.066	−.074
$P_7 = (1, 1, 1, 1, 13, 13)$	ANOVA	.066	***.043***	***.031***	.026	.056	.047	.211
	MIVQUE(0)	.033	.008	−.003	−.007	.038	.030	.245
	MIVQUE(A)	.071	***.057***	***.047***	.025	.002	−.062	−.068
	REML = I-MIVQUE	.070	***.062***	***.059***	***.062***	.075	.069	.142
	ML	.030	***−.013***	***−.044***	***−.119***	***−.207***	***−.395***	***−.838***
	ML-adj	.036	.004	−.013	−.043	−.048	−.074	−.006
$P_8 = (3, 3, 3, 5, 5, 5, 7, 7, 7)$	ANOVA	.044	***.021***	.010	.003	.004	−.014	−.019
	MIVQUE(0)	.041	***.020***	.009	.002	.006	−.012	−.018
	MIVQUE(A)	.045	***.023***	.011	.001	−.001	−.023	−.027
	REML = I-MIVQUE	.043	***.021***	.010	.003	.004	−.016	−.019
	ML	.029	−.008	***−.033***	***−.077***	***−.133***	***−.262***	***−.598***
	ML-adj	.033	.003	−.012	−.024	−.025	−.045	−.048
$P_9 = (1, 1, 1, 5, 5, 5, 9, 9, 9)$	ANOVA	.045	***.018***	.007	.003	.014	.003	.007
	MIVQUE(0)	.036	***.015***	.006	.002	.020	.006	−.001
	MIVQUE(A)	.044	***.023***	.012	.003	−.008	−.046	−.068
	REML = I-MIVQUE	.039	***.018***	.009	.009	.011	−.006	.007
	ML	.024	***−.015***	***−.040***	***−.083***	***−.141***	***−.273***	***−.598***
	ML-adj	.027	−.005	***−.020***	−.031	−.034	−.057	−.048
$P_{11} = (1, 1, 1, 1, 1, 1, 1, 19, 19)$	ANOVA	.058	***.035***	***.020***	.008	.028	.033	.113
	MIVQUE(0)	.021	−.000	−.018	−.026	−.006	.014	.131
	MIVQUE(A)	.061	***.048***	***.036***	.006	−.014	−.069	−.126
	REML = I-MIVQUE	.059	***.053***	***.049***	.042	.065	.064	.068
	ML	.029	−.007	***−.033***	***−.090***	***−.141***	***−.259***	***−.587***
	ML-adj	.033	.004	−.012	−.039	−.034	−.041	−.035
$P_{12} = (2, 10, 18)$	ANOVA	.046	***.031***	***.032***	.021	.037	.062	−.049
	MIVQUE(0)	.037	***.027***	***.031***	.023	.042	.074	−.057
	MIVQUE(A)	.052	***.037***	***.029***	.005	−.006	−.019	−.119
	REML = I-MIVQUE	.054	***.043***	***.041***	.032	.039	.046	−.027
	ML	.016	***−.038***	***−.081***	***−.200***	***−.372***	***−.707***	***−1.76***
	ML-adj	.024	−.007	***−.022***	***−.050***	−.058	−.060	−.140
$P_{13} = (3, 15, 27)$	ANOVA	.031	***.017***	.009	.004	.009	.004	−.034
	MIVQUE(0)	.025	***.014***	.009	.007	.014	−.004	−.041
	MIVQUE(A)	.034	***.021***	.006	−.017	−.033	−.029	−.085
	REML = I-MIVQUE	.035	***.027***	.016	.007	.004	.024	−.017
	ML	.010	***−.041***	***−.087***	***−.202***	***−.376***	***−.701***	***−1.73***
	ML-adj	.016	***−.012***	***−.030***	***−.053***	−.064	−.051	−.095

cussion of MSE's in Section 4.2. A better approximation for the magnitude of the downward bias is $(\sigma_e^2/N + \sigma_a^2/a)$, but σ_e^2/N is comparatively small here because $\sigma_e^2 \equiv 1$, so we have opted to use the simpler approximation σ_a^2/a.

Requiring nonnegativity induces an upward bias in all of these estimators that is most pronounced when σ_a^2/σ_e^2 is very small, which is when negative estimates of σ_a^2 are most likely to arise. For the ML estimator, this upward bias effect is lessened by the downward bias discussed in the preceding paragraph.

The ANOVA, MIVQUE(0), MIVQUE(A), and REML estimators of σ_a^2 differ very little in bias. Only for P_7 and P_{11}, the n-patterns that have been shown to be worst cases in earlier work, with σ_a^2/σ_e^2 very small, does MIVQUE(0) have less bias than the other estimators.

4.2 MSE's of Estimators of σ_a^2

Table 2 presents the MSE's of the estimators of σ_a^2, computed from the usable replicates and divided by the lower bound for quadratic unbiased estimators (QUE's). The QUE lower bounds are also shown, and are the variances of the MIVQUE estimators obtained from (6) without iteration, taking the true value of σ_a^2/σ_e^2 as the a priori input. (The variances can be calculated using equation (18) of Swallow and Searle 1978; no "data" are needed.) Employing the QUE lower bound as a scaling factor is simply a convenient way, when comparing these estimators, to deal with the strong dependence of all of these MSE's on the magnitude of σ_a^2 and on the number of subgroups a. In many cases the biased estimators being compared have MSE's smaller than the lower bound for QUE's, giving ratios less than unity. Values of the scaled MSE's that are $\leq .80$ or ≥ 1.20 are italicized in the table.

The ML estimator consistently has the smallest MSE, although its MSE superiority decreases as the number of groups increases under the same type of unbalancedness (compare P_1, P_4, and P_8, or P_2, P_5, and P_9). When $\sigma_a^2/\sigma_e^2 < 0.5$, the small bias and low MSE make the ML estimator the one of choice. However, as discussed in Section 4.1, for small a the ML estimator of σ_a^2 may have substantial downward bias when σ_a^2/σ_e^2 is large; this bias has an interesting effect on the MSE. Since the estimator is bounded below by zero, the bias concentrates the sampling distribution, reducing the variance by more than enough to offset the squared bias, so the MSE actually decreases. The greater the bias, the more the variance and MSE decrease. The bias is greatest when the number of groups is smallest, so that is when the MSE superiority of the ML estimator is most evident. To illustrate this relationship between the bias and the MSE of the ML estimator, we have added the bias-corrected ML-adj estimator in Table 2 as in Table 1. The MSE of the ML-adj estimator generally agrees with the MSE's of competing estimators. Apparently, when the bias of the ML estimator is large enough to be objectionable, bias correction results in a loss of MSE superiority; the bias-corrected estimator is as good as other estimators, but it may be more difficult to compute.

There is no apparent advantage of REML over MIVQUE(A), the first iterate of REML. In fact, under P_7 or P_{11} with σ_a^2/σ_e^2 small, iterating makes the estimates worse. Or course, MIVQUE with poorer a priori values than the ANOVA estimates would not be expected to compare as favorably with REML overall.

When $\sigma_a^2/\sigma_e^2 = 0$, MIVQUE(0) is MIVQUE with perfect a priori values. In this case MIVQUE(0) is second only to ML, outperforming even ML under n-patterns P_7 and P_{11}. When the data are mildly unbalanced (P_1, P_4, and P_8), the advantage of MIVQUE(0) over the MIVQUE(A), REML, or ANOVA estimator is slight; under more severely unbalanced data the gains with MIVQUE(0) can be sizable.

For $0 < \sigma_a^2/\sigma_e^2 < 1.0$, there is little difference between MIVQUE(0), MIVQUE(A), REML, and the ANOVA estimators, except under P_7 and P_{11}, where MIVQUE(A) and especially REML suffer. The ANOVA estimator is always reasonable in this range.

When $\sigma_a^2/\sigma_e^2 \geq 1.0$, MIVQUE(0) performs poorly, even in mildly unbalanced cases. This agrees with Swallow's (1981) conclusion, namely that when the a priori value of σ_a^2/σ_e^2 is a severe underestimate, MIVQUE does very badly. MIVQUE(0)'s poor performance when $\sigma_a^2/\sigma_e^2 \geq 1.0$ renders MIVQUE(0) an unsatisfactory default for SAS's PROCEDURE VARCOMP.

Provided $\sigma_a^2/\sigma_e^2 > 1.0$, the MSE's of the MIVQUE(A) and REML estimators approximate the QUE lower bound no matter how unbalanced the data are. The ANOVA estimator suffers under badly unbalanced data but performs very poorly only under the unlikely n-patterns P_7 and P_{11}.

The standard errors of the bias values reported in Table 1 can be determined using Table 2. For example, for P_1 and $\sigma_a^2/\sigma_e^2 = .5$, the estimated bias of the ANOVA estimator is .042 from Table 1. The estimated variance of this value is $[.980(.525) - (.042)^2]/10000 = .0000513 = (.0072)^2$, where .980 and .525 are the ratio and QUE lower bound, respectively, from Table 2.

4.3 Biases of Estimators of σ_e^2

The biases (not shown) of all of these estimators are negligible over the range of conditions studied. Even relative biases are generally under 10 percent and

Table 2. *Ratios of MSE's of Estimators of σ_a^2 to the QUE Lower Bound, and the QUE Lower Bound*

		$\sigma_a^2/\sigma_e^2 = \sigma_a^2$ ($\sigma_e^2 \equiv 1$)						
n-pattern		.0	.1	.2	.5	1.0	2.0	5.0
$P_1 = (3, 5, 7)$	ANOVA	***.70***	***.75***	.85	.98	1.03	.98	1.06
	MIVQUE(0)	***.68***	***.76***	.89	1.12	1.19	1.10	***1.21***
	MIVQUE(A)	***.74***	***.80***	.88	.95	.99	.98	1.03
	REML = I-MIVQUE	***.74***	.81	.89	.96	1.00	.99	1.03
	ML	***.21***	***.27***	***.36***	***.47***	***.53***	***.54***	***.56***
	ML-adj	***.46***	***.60***	***.75***	.90	.98	.98	1.03
	QUE Lower Bound	.049	.104	.179	.525	1.50	4.95	27.3
$P_2 = (1, 5, 9)$	ANOVA	.98	.84	.84	.94	1.12	1.15	***1.26***
	MIVQUE(0)	***.75***	.87	.96	1.16	***1.42***	***1.48***	***1.67***
	MIVQUE(A)	***1.72***	1.16	.97	.97	1.00	.93	.96
	REML = I-MIVQUE	***2.17***	***1.42***	1.15	1.12	1.11	.99	.98
	ML	***.50***	***.38***	***.38***	***.48***	***.53***	***.53***	***.55***
	ML-adj	1.12	.86	.81	.94	1.02	.96	.98
	QUE Lower Bound	.051	.125	.227	.678	1.86	5.73	29.4
$P_4 = (3, 3, 5, 5, 7, 7)$	ANOVA	***.68***	***.78***	.85	.96	1.02	1.06	1.08
	MIVQUE(0)	***.66***	.84	.94	1.15	***1.22***	***1.30***	***1.36***
	MIVQUE(A)	***.71***	***.80***	.85	.93	.98	1.02	1.00
	REML = I-MIVQUE	***.70***	.82	.87	.95	.99	1.02	1.00
	ML	***.34***	***.48***	***.58***	***.71***	***.76***	***.78***	***.76***
	ML-adj	***.49***	***.68***	***.80***	.94	.99	1.02	1.00
	QUE Lower Bound	.018	.040	.070	.208	.599	1.98	10.9
$P_5 = (1, 1, 5, 5, 9, 9)$	ANOVA	.88	***.78***	.88	1.01	1.07	***1.20***	***1.30***
	MIVQUE(0)	***.68***	.84	1.03	***1.31***	***1.44***	***1.65***	***1.84***
	MIVQUE(A)	1.02	.88	.94	.96	.95	.97	.98
	REML = I-MIVQUE	1.06	.98	1.07	1.08	1.04	1.03	.99
	ML	***.45***	***.51***	***.63***	***.73***	***.76***	***.77***	***.76***
	ML-adj	***.64***	***.73***	.88	.99	1.00	1.02	.99
	QUE Lower Bound	.015	.041	.079	.250	.711	2.25	11.7
$P_7 = (1, 1, 1, 1, 13, 13)$	ANOVA	***1.60***	.90	.90	1.14	***1.41***	***1.70***	***2.29***
	MIVQUE(0)	***.67***	.88	1.10	***1.75***	***2.38***	***3.00***	***4.09***
	MIVQUE(A)	***2.81***	***1.40***	1.17	.98	.94	.94	.98
	REML = I-MIVQUE	***4.15***	***1.97***	***1.56***	***1.23***	1.10	1.02	1.01
	ML	***1.53***	.90	.82	***.79***	***.77***	***.76***	***.76***
	ML-adj	***2.20***	***1.29***	1.16	1.08	1.05	1.01	1.01
	QUE Lower Bound	.012	.054	.115	.367	.956	2.73	12.8
$P_8 = (3, 3, 3, 5, 5, 5, 7, 7, 7)$	ANOVA	***.67***	***.76***	.91	1.01	1.06	1.05	1.05
	MIVQUE(0)	***.60***	***.79***	1.00	1.18	***1.28***	***1.30***	***1.33***
	MIVQUE(A)	***.69***	***.78***	.91	.98	1.02	1.00	.96
	REML = I-MIVQUE	***.68***	***.79***	.93	.99	1.02	1.00	.96
	ML	***.39***	***.55***	***.72***	.82	.86	.85	.82
	ML-adj	***.50***	***.70***	.88	.99	1.03	1.01	.97
	QUE Lower Bound	.011	.025	.044	.130	.374	1.24	6.83
$P_9 = (1, 1, 1, 5, 5, 5, 9, 9, 9)$	ANOVA	.88	***.79***	.90	.99	1.10	1.19	***1.32***
	MIVQUE(0)	***.64***	.84	1.05	***1.24***	***1.47***	***1.62***	***1.82***
	MIVQUE(A)	.88	.86	.92	.97	.97	.97	1.01
	REML = I-MIVQUE	.82	.92	.99	1.07	1.04	1.01	1.02
	ML	***.42***	***.60***	***.73***	.84	.85	.84	.86
	ML-adj	***.53***	***.75***	.88	1.02	1.02	1.00	1.02
	QUE Lower Bound	.009	.025	.048	.153	.440	1.40	7.30
$P_{11} = (1, 1, 1, 1, 1, 1, 1, 19, 19)$	ANOVA	***2.47***	.96	.89	1.15	***1.61***	***2.14***	***2.92***
	MIVQUE(0)	***.62***	.93	1.14	***1.90***	***3.03***	***4.19***	***5.81***
	MIVQUE(A)	***4.51***	***1.49***	1.15	.96	.90	.96	1.00
	REML = I-MIVQUE	***7.02***	***2.09***	***1.55***	1.18	1.05	1.03	1.01
	ML	***3.35***	1.15	.98	.88	.84	.85	.84
	ML-adj	***4.25***	***1.46***	***1.22***	1.08	1.03	1.03	1.01
	QUE Lower Bound	.006	.037	.085	.271	.678	1.84	8.30
$P_{12} = (2, 10, 18)$	ANOVA	***.97***	.87	.99	1.09	***1.20***	***1.36***	***1.30***
	MIVQUE(0)	***.74***	.98	***1.20***	***1.37***	***1.53***	***1.77***	***1.70***
	MIVQUE(A)	***1.54***	1.04	.96	.94	.96	1.03	.97
	REML = I-MIVQUE	***2.03***	***1.28***	1.13	1.05	1.02	1.06	.98
	ML	***.39***	***.41***	***.45***	***.51***	***.54***	***.57***	***.55***
	ML-adj	.87	.86	.89	.96	.99	1.05	.98
	QUE Lower Bound	.012	.056	.125	.461	1.43	4.87	27.2
$P_{13} = (3, 15, 27)$	ANOVA	.99	.86	.95	1.10	1.17	***1.33***	***1.38***
	MIVQUE(0)	***.77***	.98	1.15	***1.39***	***1.50***	***1.71***	***1.77***
	MIVQUE(A)	***1.58***	.96	.91	.92	.92	1.03	1.03
	REML = I-MIVQUE	***2.06***	1.16	1.04	1.00	.96	1.05	1.04
	ML	***.42***	***.42***	***.47***	***.53***	***.53***	***.57***	***.57***
	ML-adj	.93	.84	.89	.96	.95	1.05	1.03
	QUE Lower Bound	.005	.039	.096	.392	1.29	4.58	26.5

exceed 25 percent only for the ML estimator with $a \leq 6$ and $\sigma_a^2/\sigma_e^2 \leq 0.2$.

4.4 MSE's of Estimators of σ_e^2

Table 3 is similar to Table 2, except for estimating σ_e^2 ($\sigma_e^2 \equiv 1$). The QUE lower bounds are calculated using equation (17) of Swallow and Searle (1978). Values of ratio to the lower bound $\leq .80$ of ≥ 1.20 are italicized in the table.

The striking conclusion from Table 3 is that when $\sigma_a^2/\sigma_e^2 \geq 1.0$, MIVQUE(0) can be terrible, even for mildly unbalanced data. For $\sigma_a^2/\sigma_e^2 > 5.0$ (not shown), MIVQUE(0)'s problems become even more pronounced. At $\sigma_a^2/\sigma_e^2 = 20.0$, the ratio of the MIVQUE(0) MSE to the QUE lower bound is 76.1 under P_1, 139 under P_4, 154 under P_8, and 656 under P_{11}; comparable values for all other estimators are near unity.

There is very little difference between the MIVQUE(A), REML, ML, and ANOVA estimators of σ_e^2. All perform well over the broad range of conditions studied. When σ_a^2/σ_e^2 and the number of groups a are small, ML has a somewhat smaller MSE; these are the same cases for which the downward bias of the ML estimator is greatest. As discussed for the ML estimator of σ_a^2, downward bias reduces variance and thereby MSE, since the increase in squared bias is less than the decrease in variance.

Increasing the total number of observations N, either by increasing group sizes (compare P_2, P_{12}, P_{13}) or by adding more groups (P_2, P_5, P_9), benefits all estimators, as seen in the QUE lower bounds. However, compared with the other estimators, MIVQUE(0) becomes an even poorer choice as N increases.

As mentioned earlier, three n-patterns used in Swallow and Searle (1978) and Swallow (1981) are not included in the tables presented here: $P_3 = (1, 7, 7)$, $P_6 = (1, 1, 7, 7, 7, 7)$, and $P_{10} = (1, 1, 1, 7, 7, 7, 7, 7, 7)$. The results for P_3, P_6, and P_{10} were similar to those reported for P_2, P_5, and P_9, respectively, in Tables 1–3.

4.5 Effect of Starting Point for Iteration, and Convergence Rates

For both ML and REML we used the ratio of ANOVA estimates ($\hat{\sigma}_a^2/\hat{\sigma}_e^2$) as the starting point for iteration in Tables 1–3. To check the dependence of these estimators on starting point, we tested also the ANOVA ratio divided by 10 and the ANOVA ratio times 10 for P_2 as an example. These three starting points were compared for exactly the same replicates. Some of the results of this comparison are shown in Table 4.

As Table 4 indicates, the median number of iterations to convergence for either ML or REML has little dependence on starting point. Both estimators typically converge in two to four iterations when σ_a^2/σ_e^2 is small, and in four to six when σ_a^2/σ_e^2 is large.

When σ_a^2/σ_e^2 is small, ML has fewer replicates requiring 20 or more iterations to converge than does REML; when σ_a^2/σ_e^2 is large, REML has fewer cases of slow convergence than ML. In general, ML and REML exhibit slow convergence on different replicates. For small σ_a^2/σ_e^2, the number of slow convergences is slightly reduced by a smaller starting point for iteration and increased by a larger one, especially for ML.

The MSE's and biases of the ML and REML estimators of σ_a^2 and σ_e^2 are affected negligibly by starting point. The values given for P_2 in Tables 1–3 could as well have been for starting iteration at a tenth or ten times the ANOVA ratio.

As noted earlier, when either ML or REML had failed to converge in 20 iterations for a particular replication, that replication was dropped for all estimators. For mildly unbalanced n-patterns (P_1, P_4, P_8), this occurred in about .1% of the replicates; for severely unbalanced n-patterns (P_2, P_5, P_9, P_{12}, P_{13}, and P_3, P_6, P_{10}), it occurred in roughly 1%, with P_2 having the most problems; and for the worst-case n-patterns (P_7, P_{11}), it occurred in 10% of the replicates. By enforcing nonnegativity on all estimators of σ_a^2, including after each iteration for ML and REML, we seemed to avoid all computing problems except occasional slow convergence.

5. SUMMARY AND CONCLUSIONS

1. The ML estimator of σ_a^2 has downward bias which may be large when $\sigma_a^2/\sigma_e^2 \geq .5$. It also has smallest MSE, but that is in part a by-product of the bias that has the effect of reducing the variance and thus the MSE. When $\sigma_a^2/\sigma_e^2 < .5$, the ML estimator of σ_a^2 has small bias, low MSE, and is the preferred estimator; for $\sigma_a^2/\sigma_e^2 \geq .5$, the bias may be objectionable, and bias correction eliminates the MSE superiority over the ANOVA, MIVQUE(A), and REML estimators, which usually differ little among themselves.

2. For estimating σ_e^2, the five estimators have negligible bias and, except for MIVQUE(0), comparable MSE's.

3. MIVQUE(0) assumes importance through the wide exposure it receives as the default in SAS's PROCEDURE VARCOMP. For estimating σ_a^2, MIVQUE(0) performs well when $\sigma_a^2 \approx 0$, as it should, but not as well as the ML estimator; when $\sigma_a^2/\sigma_e^2 = 0.1$, it is no better than the ANOVA estimator. When $\sigma_a^2/\sigma_e^2 \geq 1.0$, the MIVQUE(0) is a poor estimator for σ_a^2 and a terrible estimator for σ_e^2, even for only mildly unbalanced data. MIVQUE(0) is a dangerous default and should be used only when one is confident that $\sigma_a^2 \approx 0$.

Table 3. *Ratios of MSE's of Estimators of σ_e^2 to the QUE Lower Bound, and the QUE Lower Bound*

		$\sigma_a^2/\sigma_e^2 = \sigma_a^2$ ($\sigma_e^2 \equiv 1$)						
n-pattern		.0	.1	.2	.5	1.0	2.0	5.0
P_1 = (3, 5, 7)	ANOVA	.98	1.03	1.00	1.02	1.00	1.01	1.00
	MIVQUE(0)	.96	1.02	1.01	1.10	***1.26***	***1.90***	***5.86***
	MIVQUE(A)	.96	1.01	.99	1.02	1.00	1.01	1.00
	REML = I-MIVQUE	.96	1.01	.99	1.02	1.00	1.01	1.00
	ML	***.78***	.84	.85	.91	.92	.98	.98
	QUE Lower Bound	.165	.166	.166	.166	.167	.167	.167
P_2 = (1, 5, 9)	ANOVA	1.05	1.04	1.05	1.02	1.01	.99	1.00
	MIVQUE(0)	1.00	1.01	1.03	1.07	***1.27***	***1.83***	***5.73***
	MIVQUE(A)	1.00	1.00	1.01	.99	.99	.99	1.00
	REML = I-MIVQUE	1.01	1.01	1.02	1.00	1.00	.99	1.00
	ML	.83	.86	.89	.89	.94	.96	.99
	QUE Lower Bound	.159	.161	.162	.164	.166	.166	.167
P_4 = (3, 3, 5, 5, 7, 7)	ANOVA	1.01	1.02	1.00	1.00	1.00	.97	.99
	MIVQUE(0)	1.00	1.02	1.02	1.15	***1.45***	***2.63***	***9.84***
	MIVQUE(A)	1.00	1.01	.98	.99	1.00	.97	1.00
	REML = I-MIVQUE	1.00	1.01	.99	1.00	1.00	.97	.99
	ML	.84	.88	.90	.96	.99	.97	1.00
	QUE Lower Bound	.082	.083	.083	.083	.083	.083	.083
P_5 = (1, 1, 5, 5, 9, 9)	ANOVA	1.07	1.02	1.02	1.05	1.02	1.00	1.03
	MIVQUE(0)	1.01	1.00	1.03	***1.24***	***1.70***	***3.57***	***15.4***
	MIVQUE(A)	1.01	.97	.98	1.02	1.00	1.00	1.05
	REML = I-MIVQUE	1.01	.98	.99	1.04	1.02	1.01	1.03
	ML	.87	.89	.91	.99	1.00	1.01	1.04
	QUE Lower Bound	.078	.080	.080	.082	.083	.083	.083
P_7 = (1, 1, 1, 1, 13, 13)	ANOVA	1.13	1.06	1.06	.98	.99	.98	1.00
	MIVQUE(0)	1.01	.98	1.07	***1.29***	***2.13***	***5.36***	***26.5***
	MIVQUE(A)	1.02	.97	1.00	.95	.99	1.00	1.04
	REML = I-MIVQUE	1.03	.99	1.02	.97	1.01	1.00	1.01
	ML	.92	.91	.94	.93	.99	1.00	1.03
	QUE Lower Bound	.074	.076	.078	.081	.082	.083	.083
P_8 = (3, 3, 3, 5, 5, 5, 7, 7, 7)	ANOVA	1.02	.99	.99	.99	.98	1.01	1.01
	MIVQUE(0)	1.00	.99	1.02	1.14	***1.45***	***2.74***	***11.2***
	MIVQUE(A)	1.00	.98	.98	.98	.98	1.01	1.01
	REML = I-MIVQUE	1.00	.98	.99	.99	.98	1.01	1.01
	ML	.85	.88	.93	.97	.98	1.01	1.01
	QUE Lower Bound	.054	.055	.055	.055	.056	.056	.056
P_9 = (1, 1, 1, 5, 5, 5, 9, 9, 9)	ANOVA	1.08	1.07	1.07	1.05	1.01	.99	1.02
	MIVQUE(0)	1.01	1.03	1.07	***1.25***	***1.75***	***3.63***	***16.0***
	MIVQUE(A)	1.01	1.01	1.02	1.02	.99	1.00	1.03
	REML = I-MIVQUE	1.01	1.02	1.03	1.03	1.00	1.00	1.02
	ML	.88	.94	.98	1.01	1.00	1.00	1.03
	QUE Lower Bound	.052	.053	.054	.054	.055	.055	.056
P_{11} = (1, 1, 1, 1, 1, 1, 1, 19, 19)	ANOVA	1.16	1.10	1.08	1.01	.98	1.01	.99
	MIVQUE(0)	.99	1.02	1.08	***1.48***	***2.98***	***8.36***	***45.6***
	MIVQUE(A)	1.01	1.00	.99	.97	.98	1.04	1.03
	REML = I-MIVQUE	1.03	1.02	1.02	1.00	1.00	1.03	.99
	ML	.95	.96	.96	.98	1.00	1.04	1.00
	QUE Lower Bound	.048	.050	.051	.054	.055	.055	.055
P_{12} = (2, 10, 18)	ANOVA	1.04	1.01	1.00	1.01	1.00	1.03	.99
	MIVQUE(0)	1.01	1.00	1.02	1.14	***1.43***	***2.63***	***9.23***
	MIVQUE(A)	1.01	.99	.99	1.00	1.00	1.03	1.00
	REML = I-MIVQUE	1.01	.99	.99	1.00	1.00	1.03	.99
	ML	.93	.92	.94	.97	.98	1.02	.99
	QUE Lower Bound	.072	.073	.074	.074	.074	.074	.074
P_{13} = (3, 15, 27)	ANOVA	1.01	1.00	1.02	.99	.99	.99	1.00
	MIVQUE(0)	.99	.99	1.04	1.14	***1.52***	***3.17***	***12.9***
	MIVQUE(A)	.99	.99	1.01	.99	.99	.99	1.00
	REML = I-MIVQUE	.99	.99	1.01	.99	.99	.99	1.00
	ML	.94	.95	.98	.98	.98	.99	1.00
	QUE Lower Bound	.047	.047	.047	.048	.048	.048	.048

Table 4. *Comparison of Starting Points for Iteration with REML and ML using $P_2 = (1, 5, 9)$ and Starting Points (ANOVA ratio)/10, (ANOVA ratio), and (ANOVA ratio) * 10*

	$\sigma_a^2/\sigma_e^2 = \sigma_a^2$ ($\sigma_e^2 \equiv 1$)						
	.0	.1	.2	.5	1.0	2.0	5.0
Median Number Iterations to Convergence							
REML							
(ANOVA ratio)/10	2	3	4	6	6	6	6
(ANOVA ratio)	2	3	4	5	5	5	5
(ANOVA ratio) * 10	2	4	5	6	6	6	5
ML							
(ANOVA ratio)/10	2	3	4	4	4	5	5
(ANOVA ratio)	2	3	3	4	5	5	5
(ANOVA ratio) * 10	2	3	4	5	5	6	5
Number of the 10,000 Replications Not Converged in <20 Iterations							
REML							
(ANOVA ratio)/10	81	103	95	85	81	82	59
(ANOVA ratio)	96	91	64	67	73	47	33
(ANOVA ratio) * 10	127	148	115	114	123	78	60
ML							
(ANOVA ratio)/10	51	61	83	84	117	142	110
(ANOVA ratio)	80	73	118	118	151	138	114
(ANOVA ratio) * 10	96	97	138	159	183	187	146

4. Unless the data are severely unbalanced and $\sigma_a^2/\sigma_e^2 > 1$, the ANOVA estimators are adequate. They have the added advantages of being familiar and easy to compute.

5. MIVQUE(A) is not improved by iterating to the REML estimates. However, MIVQUE(A) requires that one first calculate the ANOVA estimates, whereas the REML estimates can be obtained by using any convenient starting point for iteration (for example, setting starting values for all variance components to unity).

6. ML and REML usually converge rapidly. Varying the iteration starting point slightly affects the rate of convergence, but not the MSE's or biases of the estimators. Enforcing nonnegativity after each iteration eliminates computing problems experienced occasionally otherwise.

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